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THE

DIFFERENTIAL METHOD:

O R, A

TREATISE

CONCERNING

SUMMATION and INTERPOLATION

O F

INFINITE SERIES.

By *JAMES STIRLING*, Efq; F. R. S.

Tranflated into *Englifh*, with the Author's Approbation,

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THE

Tranſlator to the READER.

IT is needleſs for me to ſay any thing in commendation of the celebrated author's treatiſe, the character of which is ſo well eſtabliſhed, and its general method for ſumming and interpolating ſeries ſo clear and elegant, that it is deſervedly eſteemed one of the beſt performances of its kind.

I had deſigned at firſt to inſert the tranſlation in ſmall parts from time to time in the MISCELLANEA CURIOSA MATHEMATICA, now publiſhing in numbers by Mr CAVE at St John's Gate, London, for the benefit of the Engliſh reader ; but I was diverted from my reſolution by ſeveral eminent mathematicians, who deſired that I would print it at once by itſelf, as ſuch an excellent piece deſerves.

And I here look upon it as a duty incumbent on me, ingenuouſly and with gratitude to confeſs, that whatever advantages may accrue to myſelf or the publick by this Engliſh edition, I owe chiefly to my much honoured and eſteemed friend, the univerſally learned mathematician Mr Emerſon, who not only aſſiſted me in reviſing my tranſlation, but gave me the firſt hint of making it public in this way.

Being unwilling to commit any miſtakes, that might be injurious to my readers, I had recourſe to Mr Stirling himſelf, and he was ſo obliging as to give me all the aſſiſtance I needed on this occaſion, and particularly his correction and amendment to Ex. 2. p. 132, concerning which I had ſome ſuſpicion.

Some mathematicians in London objected to me, that Mr Stirling had no where ſhown from a given ſeries how to find the equations of the ſucceſſive terms ; but they are miſtaken ; for he has ſhown this in the fourth and fifth pages. And tho' it requires perhaps a little explanation, yet for the moſt part it appears by inſpection, if attentively regarded ; which was the reaſon, I ſuppoſe, that he had given no particular rule for it.

In a postscript of a letter that I had the honour to receive from Mr Stirling, dated the 18th of May 1747, in answer to this objection, " I had (says he)
" almost forgot to give an answer to that part of your obliging letter,
" where you say that some eminent mathematicians had observed to you,
" that I had no where shown from a given series how to find the equation of
" the successive terms : To which I answer, that the thing is self evident,
" or very easy in such series as I proposed to sum or interpole ; but in many
" kinds of series the equation to the terms cannot be found, at least if their
" relation be considered in the manner that I have done. For instance,
" suppose a trinomial series, A, B, C, D, &c. where the relations of the
" terms are

$$p\mathrm{C}=q\mathrm{B}+r\mathrm{A}$$
$$p'\mathrm{D}=q'\mathrm{C}+r'\mathrm{B}$$
$$p''\mathrm{E}=q''\mathrm{D}+r''\mathrm{C}$$
$$p'''\mathrm{F}=q'''\mathrm{E}+r'''\mathrm{D}$$
$$\&c.$$

" Now if these three progressions can be carried on ad libitum, I mean
" p, p', p'', p''', &c. q, q', q'', q''', &c. and r, r', r'', r''' &c. then
" the series is given ; but nevertheless an equation to the successive terms
" of A, B, C, D, &c. cannot be found, unless the terms of the three pro-
" gressions be assignable. In the series which I have consider'd, the terms
" p, p', p'', and q, q', q'', &c. and r, r', r'', &c. are generally to be
" assigned by bare inspection ; and, if not, then Newton's differential
" method does it ; for they are always to be assigned by a quantity of this
" form, $a+bz+cz^2+dz^3+$ &c. where z can never rise above a deter-
" minate dimension. For this reason I gave no rule for finding the equa-
" tions to the terms of the series which I considered. And I did not pro-
" pose to solve the problem generally, because I knew it was not to be done."
Thus far Mr Stirling.

*As for my part, if what I have here done meet with a favourable recep-
tion, I may perhaps publish in our own language some other valuable parts of
this celebrated author's works, since I have his consent to proceed.*

Haughton Park, *near* Retford,   F. HOLLIDAY.
Nottinghamshire, July 14, 1748.

THE

PREFACE.

WHEREAS it very often happens, that series converge to the truth so slowly, that they conduce no more to the end proposed, than if they were really diverging; to remedy this inconvenience, I have, in the first part of the ensuing treatise, exhibited several theorems whereby we readily obtain the values of the most slowly converging series; in which my design was to shew that those problems which depend on quadratures may admit of as just a solution, as those which are reduced to adfected equations. For I do not here enquire what series are summable (as some of my readers may perhaps conjecture from the title page) but how to obtain the values of those series which cannot be summed.

Mr James Gregory *published a small treatise of this nature in his Appendix to the true quadrature of the circle and hyperbola, where he has shown a very easy method of approximating to the areas of these curves from a very few given* polygons, *of no great number of sides; and has thereby render'd* Archimedes's *Method of Exhaustions so easy, that if the same could be extended to other curves with success, it would be needless to spend any more labour in investigating their areas; and what Mr* Gregory *hath there effected in a series of* polygons, *we have here illustrated almost universally, in all other series, where there is a simple relation of terms.*

The great Sir Isaac Newton *had formerly spent some thoughts on this subject, as appears from his first letter to M.* Oldenburgh, *printed in* Collins's Commercium epistolicum; *where, after he had brought out, by computation, the area of the circle to sixteen places of decimals,* "If *(says* " he) I had apply'd other methods, I could, by the same number of terms " of the series, have obtain'd many more places, suppose twenty five or " more; but I had only here a mind to shew what might be done by a sim- " ple computation of series."

I do not find, however, amongst his writings (at least hitherto published) the least hint, from whence we can form any conjecture of these methods;

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tho' he had the fairest opportunity of shewing them in this very letter. He likewise further observes, in another letter to the same gentleman, printed in the same book, that he had found out some things relating to reduction of infinite series into finite ones, where the nature of the things would admit. If these discoveries should be published in his posthumous works, they will without doubt much illustrate this doctrine; for general theorems, which exhibit the values of series accurately, when it can be done, will necessarily approximate, in other cases, if they be rightly applied.

The principle, commonly apply'd to this purpose, is the assumption of the difference of two successive values of any quantity, that terms may from thence be formed, whose sum was before known; namely, the same principle which Newton formerly made use of, to obtain the ordinate of a curve from the given area. Tho' this may be universal in quadratures, yet it is only particular in summations, since it is applicable only to those series, whose terms can be assign'd; whereas the assignation of the sum, or of any term, is equally easy in such series as generally use to arise from the quadratures of figures.

Sir Isaac Newton's Differential Method hath supply'd us with a far more general foundation; for he describes a parabolic curve through the extremities of any number of ordinates or terms, and thereby assigns the value of every intermediate term by an infinite series, which nevertheless will not approach to the truth, if the term be very remote from the beginning. That I might obtain, therefore, the remotest terms of the series, I described a hyperbolic figure through the extremities of the terms; and the thing succeeded, the value of the term, tho' never so remote, coming out by converging series. The general solution of this problem includes a very obvious and easy case, namely, the finding of a term at an infinite distance from the first, which is indeed equivalent to the summation of the series. But the description of every single geometrical curve through given points only suffices for one kind of series; and there are innumerable other kinds, which can by no means be had from this foundation. For the value of a term, found either by the parabola or hyperbola, does not approximate, except when the differences, taken according to the Newtonian rules, constitute a progression decreasing sufficiently swift.

After I had made these discoveries, I began to consider the relation of the terms, the most remarkable and simple property of the series, which commonly uses to be applied for their continuation. For I was not ignorant that M. De Moivre had introduc'd this property of the terms into his Algebra,
with

with the greatest success, as the foundation for solving the most difficult problems about recurring series. I therefore determined to try whether it could be also extended to other series, which I indeed doubted, as there is so great a difference between recurring and other series; however, upon trial, the thing succeeded beyond my expectation, for I soon discovered that this invention of M. De Moivre contained the most general principles, and likewise the most simple ones, not only in recurring series, but in all others wherein the relation of the terms is varied according to some regular law; for though the relation of the terms be variable, it is however easily assignable, and from thence summations and interpolations, and other such like difficult problems are brought to a certain kind of equations, which, besides the root to be extracted, involve other unknown quantities that cannot be expunged; notwithstanding which, the resolution of these equations is sometimes effected with extraordinary facility on other occasions, tho' it does not so well succeed, but by the help of M. De Moivre's method concerning the assignation of terms in recurring series, which is almost the only subject treated of in the following sheets.

The problem for finding a middle uncia in a very high power of a binomial was solved by M. De Moivre, some years before I meddled with it; nor is it probable I should, to this day, have thought of it, had not my much esteemed friend, Sir Alexander Cuming, Bart. given me an occasion, by intimating a strong suspicion whether this problem could be solved by Sir Isaac Newton's Differential Method, or not.

E R R A T A.

PAGE 17, Line 30, *for* is drawn, *read* is to be drawn ; *Page* 21,
L. 22, *for* $\dfrac{a+bz^2+cz^3+dz^4,\ \&c.}{z.z+a.z+b.z+c.z+d,\&c.}$, *read* $\dfrac{A+Bz^2+Cz^3+Dz^4,\ \&c.}{z.z+a.z+b.z+c.z+d,\ \&c.}$
Page 27, *laſt Line, for* $1-t$, *read* $t-1$. *Page* 34, L. 2, *for* m S, *read*
m S'. *Line* 13, *for* $\dfrac{Ax^o}{z}$, *read* $\dfrac{Ax^o}{1^x}$. *Page* 38, L. 5, *dele* ag. *Page* 43,
Line 13, *for* $\overline{z+1}$ the equation, *read* $\overline{z+1}$ be the equation. L. 19, *for*
$\dfrac{1}{zz+3}$, *read* $\dfrac{1}{2z+3}$. *Page* 49, L. 3, *read* compared. *Page* 50, *Ex.* IV.
Line 1, *read* Let this ſeries $1+\dfrac{1.1}{2.5}A+\dfrac{3.5}{4.9}B+\dfrac{5.9}{6.13}C+\dfrac{7.13}{8.17}D+\dfrac{9.17}{10.21}E$
$+$ &c. *Page* 56, *Line* 11, *read* $4z+8$. *Page* 62, *Line* 11, *dele* $+t$ S^{II}.
P. 66, L.8, *read* $11784x^7$. L. 11, *after* and, *read* therefore. P.72, L.1,
read flowing equation. P.73, L. 6, *read* thereof. P.74, L. 8, *read* $720r^6$.
Page 81, *Line* 24, *read* $\dfrac{r+3}{p+3}$D. *Page* 82, *Line* 19, *read* G$=1\times\dfrac{r}{r+6}$.
Page 84, *laſt Line but one, read* $p=\frac{1}{1}$. *Page* 85, *and two firſt Lines of*
Page 86, *the reader is deſired to blot out, occaſioned by the careleſsneſs of*
the compoſer. *Page* 90, *Line* 28, *read* antecedents. *Page* 92, *Line* 4,
read $92z+9z^2$. *Page* 107, *laſt but one, read* $\dfrac{m-2}{m+2}$. *Page* 118, L. 22,
read area. *Read* P. 125, 126, 127, 128. *Page* 130, L. 22, *read* pro-
poſition *for* problem. *Page* 135, L. 30, *read* $\frac{3}{4}$. *Page* 137, L. 17, *read*
will be middle ordinates. *Page* 139, *fifth Line from Bottom, read* 5A
$-$9B$+$5C$-$.

A TREATISE OF THE
SUMMATION and INTERPOLATION
OF
INFINITE SERIES.

The INTRODUCTION.

AS curves are not determined from any of their given ordinates, but from the relation in general between their abscissas and ordinates, so neither are series determined from any of their given terms, but from the relation between their successive terms. For quantities, how many soever, when finite, may constitute terms in diverse series; but the series is singular in which the initial terms are the same, and the law is the same for forming the rest of the terms, *in infinitum.* Therefore the relations of the terms must, in the first place, be sought; and when they are found, they must be denoted by differential equations, as *Des Cartes* defined curves by algebraic equations; which being had, problems about summation and interpolation, and other such like problems belonging to series, will be solved by an *analysis* no less certain than common algebra.

Of the Relation of the TERMS.

THE terms of a series, taken two by two, three by three, &c. do, in a great measure, obtain a certain simple and obvious relation among themselves, by which the series are determined, and are produced at pleasure. For instance, if unity be divided by $1-x$, the quotient will be a geometrical progression wherein every succeeding term is to the term immediately preceding as x is to unity. For by this property, the series $1+x+x^2+x^3+x^4+x^5+$ &c. is distinguished from any other, and produced *in infinitum.*

EXAMPLE.

Let this fraction $\dfrac{1}{r+sx+tx^2}$ be proposed to be resolved into a series;

suppose $y=\dfrac{1}{r+sx+tx^2}$; and, by drawing each part into the denominator, it

will

will be $y \times \overline{r + sx + txx} = 1$, or $y \times \overline{r + sx + tx^2} - 1 = 0$. For y substitute a series of a due form, as $A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ &c. and there will result

$$\left. \begin{array}{l} rA + rB \\ -1 + sA \end{array} \right\} x \begin{array}{l} +rC \\ + sB \\ + tA \end{array} \right\} x^2 \begin{array}{l} +rD \\ + sC \\ + tB \end{array} \right\} x^3 \begin{array}{l} +rE \\ + sD \\ + tC \end{array} \right\} x^4 \begin{array}{l} +rF \\ + sE \\ + tD \end{array} \right\} x^5 + \&c. = 0;$$

where, by putting the homologous members equal to nothing, to determine the assumed coefficients, it will be $rA - 1 = 0$, $rB + sA = 0$; then $rC + sB + tA = 0$, $rD + sC + tB = 0$, $rE + sD + tC = 0$; and so on *in infinitum*. From which it appears, that there comes out the same relation between any three successive terms. Likewise for y, using a series of this form $Ax^{-2} + Bx^{-3} + Cx^{-4} + Dx^{-5} + $ &c. and by substituting the same in the equation, there will result

$$\left. \begin{array}{l} tA + tB \\ -1 + sA \end{array} \right\} x^{-1} \begin{array}{l} +tC \\ + sB \\ + rA \end{array} \right\} x^{-2} \begin{array}{l} +tD \\ + sC \\ + rB \end{array} \right\} x^{-3} \begin{array}{l} +tE \\ + sD \\ + rC \end{array} \right\} x^{-4} + \&c. = 0.$$

And now for determining the coefficients, we have these equations; $tA - 1 = 0$, $tB + sA = 0$; also $rA + sB + tC = 0$, $rB + sC + tD = 0$, $rC + sD + tE = 0$, &c. whence it appears that the same relations are produced as in the above *computus*, taking the coefficients in an inverse order. And the quantities r, s, t, which shew the relation of the terms, are the same as those in the denominator of the fraction. This property, howsoever obvious it may be, M. *De Moivre* was the first that applied it to use, in the solution of problems about infinite series, which otherwise would have been very intricate.

But in most series the relation of the terms is not constant as in those produced by division; but it is varied very often, according to a known law evident at sight; as those series which are commonly produced for quadratures, and innumerable others. For instance, in this series $1 + \frac{1}{3}x + \frac{1}{3 \cdot 5}x^2 + \frac{1}{1 \cdot 3 \cdot 5}x^3 + \frac{1}{1 \cdot 3 \cdot 5}x^4 + $ &c. the terms are continued *in infinitum*, by the continual multiplication of these fractions $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{8}{9}$, &c. and in this $1 + \frac{1}{6}x + \frac{1}{2 \cdot 6}x^2 + \frac{1}{1 \cdot 3 \cdot 5}x^3 + \frac{1}{1 \cdot 3 \cdot 5}x^4 + $ &c. by the continual multiplication of these fractions $\frac{1 \cdot 1}{2 \cdot 3}$, $\frac{3 \cdot 3}{4 \cdot 5}$, $\frac{5 \cdot 5}{6 \cdot 7}$, $\frac{7 \cdot 7}{8 \cdot 9}$, &c. and these fractions are varied according to a law manifest to any body; and consequently there will be no difficulty in their assignation.

Of *differential* EQUATIONS *which define* SERIES.

AN equation defining a series, is that which assigns in general the relation of the terms, by means of their given distances from the beginning. And the terms must be imagin'd to stand upon a right line, given in position,

tion, as so many ordinates, whose common distance is unity. Where note, for the sake of simplicity, I every where use in the following sheets unity for the common space: which I note here once for all.

The first terms of the series I denote by the first letters of the alphabet A, B, C, D, &c. A is the first, B the second, C the third, &c. And every term in general I denote by the letter T, and the rest succeeding in order by the same letter, adjoining the Roman characters I, II, III, IV, V, VI, VII, &c. for distinction's sake. Thus, if T be the tenth, T′ will be the eleventh, T″ the twelfth, T‴ the thirteenth, &c. And, in general, whatever term is defined by T, the succeeding terms will be defined universally by T′, T″, T‴, T^{iv}, &c.

The distance of the term T from any given term, or from any given point in the middle between any two terms, I denote by the indeterminate quantity z; whereby the distances of the terms T′, T″, T‴, &c. from the aforesaid term or point, will be $z+1$, $z+2$, $z+3$, &c. For the increment of the abscissa z is equal to the common interval of the terms standing upon the abscissa; and the quantities z, $z+1$, $z+2$, $z+3$, &c. mutually succeed one another, whilst the latter terms succeed the former.

These things being premis'd, let this series 1, $\frac{1}{1}x$, $\frac{1}{2}x^2$, $\frac{1}{6}x^3$, $\frac{1}{12}x^4$, $\frac{3}{4}x^5$, $\frac{6}{12}x^6$, &c. be proposed, where the relations of the terms are $B=\frac{1}{1}Ax$, $C=\frac{1}{2}Bx$, $D=\frac{1}{3}Cx$, $E=\frac{1}{4}Dx$, &c. the relation in general will be defined by this equation $T'=\frac{z+\frac{1}{2}}{z+1}Tx$; where z denotes the distance of T from the first term of the series. For by writing 0, 1, 2, 3, 4, &c. successively for z, there will be produced the relations of the terms in the proposed series. Likewise if z denote the distance of T from the second term of the series, the equation will be $T'=\frac{z+\frac{1}{2}}{z+2}Tx$, as will appear by writing the numbers -1, 0, 1, 2, 3, &c. successively for z. Or if the indeterminate quantity z denote the place of the term T in the series, its successive values will be $1,2,3,4$, &c. and the equation will be $T'=\frac{z-\frac{1}{2}}{z}$ Tx; as will appear evident to any that will try it.

Therefore innumerable diverse differential equations may define the same series, as the beginning of the abscissa z is taken in this or that point. And, on the contrary, the same equation defines innumerable diverse series, by applying diverse successive values for z. For in the equation, $T'=\frac{z-\frac{1}{2}}{z}Tx$, which defines the series now treated of, when $1, 2, 3, 4$, &c. are the values of the abscissa succeeding one another in order; write

I $\frac{1}{2}$,

$1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. fucceffively for z, and there will come out the re-
lations of the terms, *viz.* $B = \frac{2}{3}Ax$, $C = \frac{4}{5}Bx$, $D = \frac{6}{7}Cx$, &c. whence
the feries becomes A, $\frac{2}{3}Ax$, $\frac{8}{15}Ax^2$, $\frac{16}{35}Ax^3$, $\frac{128}{315}Ax^4$, &c. which is dif-
ferent from the former. But the equation always determines the feries
from given values of the abfciffa together with the firft term, when the
equation involves only two terms of the feries; as, in the laft, are given
all the terms from A, the firft being given. Neverthelefs when the e-
quation involves three terms, two muft be given; and when it involves
four, three muft be given, &c. to determine the feries.

Let now this feries x, $\frac{1}{6}x^3$, $\frac{3}{40}x^5$, $\frac{5}{112}x^7$, $\frac{35}{1152}x^9$ be propofed, where
the relations of the terms are $B = \frac{1-1}{2\cdot3}Ax^2$, $C = \frac{3\cdot1}{4\cdot4}Bx^2$, $D = \frac{5\cdot3}{6\cdot7}Cx^2$, &c.
The equation for it will be $T' = \frac{\overline{2z-1} \times \overline{2z-1}}{2z \cdot \overline{2z+1}} Tx^2$, or $T' = \frac{4zz - 4z + 1}{4zz + 2z}$

Tx^2, where the fucceffive values of z are 1, 2, 3, 4, &c. Therefore, in
the equation defining the feries, the abfciffa z may be of one, two, or
more dimenfions.

Series whofe terms are affignable, may be defined by equations affign-
ing the terms. Thus, the feries $1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - $ &c. is defin-
ed by this equation $T = \frac{-x^z}{z+1}$, as will appear by fubftituting 0, 1, 2, 3.
&c. for z. And after the fame manner the feries $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4$
$+ \frac{1}{25}x^5 + $ &c. is denoted by this equation $T = \frac{x^z}{zz}$. Such equations may
always be reduc'd to thofe of another kind; for when the terms are
affignable, their relations will likewife be affignable; and the difference
between thefe and the others will feldom be fo great, but every one may
fafely proceed as he fhall fee proper.

Hitherto it appears that the relations of the confequent terms are deriv-
ed from the antecedent, by writing for z its fucceffive value $z + 1$ in the
differential equation. Let this equation $T' = \frac{z+n}{z}T$ be propofed; write
$z + 1$ for z, T' for T, and T'' for T', and there will arife $T'' = \frac{z+n+1}{z+1}$
T', which is the relation between the terms T' and T''. In this laft
write the confequent values of the variable ones, $z + 1$, T'', and T''', for
the antecedent z, T' and T'', and you will obtain $T''' = \frac{z+n+2}{z+2}T''$, the
relation between T'' and T'''.

But, by a contrary operation, we can go back to the relation of the an-
tecedent terms, from that given relation of the confequent. Let the
equation

equation be $T'' = \dfrac{z^3 - 1}{z^3 + 3z^2 + 3z + 2} T'$; now if you will write T for T', T' for T'', and $z-1$ for z, you will have $T' = \dfrac{zz - 2z}{z^3 + 1} T$. In this manner, by going backward and forward, feries may thence be continued *in infinitum*, when their nature will admit of it: and likewife if we be ignorant what terms are denoted by T, T', T'', &c. we can inftitute a *computus* from thofe, as if they were really known.

The equations of which we have hitherto treated, involve only two terms of a feries; but they may involve more terms, and the terms, as well as the indeterminate quantity z, may be of more dimenfions; but in this fpecimen I have only treated of thofe that are more fimple.

Of the Form and Reduction of SERIES.

AFTER we have brought the feries to differential equations, we muft fhew how they are to be refolved in numbers. For it is the bufinefs of the analyft to bring out the quantities determin'd after any manner, either accurately, or as near as poffible. The roots of differential equations are very commodioufly refolved into feries of the following forms:

$$A + Bz + Cz \cdot \overline{z-1} + Dz \cdot \overline{z-1} \cdot \overline{z-2} + Ez \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + \&c.$$
$$A + \frac{B}{z} + \frac{C}{z \cdot z+1} + \frac{D}{z \cdot z+1 \cdot z+2} + \frac{E}{z \cdot z+1 \cdot z+2 \cdot z+3} + \&c.$$

Wherefore, when z is a fmall quantity, the firft form muft be ufed; but the latter, when great. And thefe feries, which are compofed of factors in arithmetical progreffion, are by far more fit for this bufinefs than the common ones, which are compofed of the powers of the indeterminate quantity, whether afcending or defcending. Moreover the latter form has this advantage, that z therein may be as great as you will, which makes the feries converge very fwift.

But if the forms of thefe feries be changed by any operations, they muft be reduc'd to the fame as at firft, that the terms may become homologous, and their *collation*, or comparifon, be inftituted as the thing requires. Suppofe this equation,

$$T = A + Bz + Cz \cdot \overline{z-1} + Dz \cdot \overline{z-1} \cdot \overline{z-2} + Ez \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + \&c.$$

the fame drawn into z will lofe its former form, and put on this new one, it becoming

$$Tz = Az + Bz^2 + Cz^2 \cdot \overline{z-1} + Dz^3 \cdot \overline{z-1} \cdot \overline{z-2} + Ez^2 \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + \&c.$$

whence it appears that its terms cannot be compared with the correfpondent ones, in the firft feries. Wherefore, to reftore its due form, I work thus:

C

$$A z - - - - - - - = A z$$
$$B z^2 - - - - - - - = B z + B z . \overline{z - 1}$$
$$C z^2 . \overline{z - 1} - - - - - = * \quad 2 C z . \overline{z - 1} + C z . \overline{z - 1} . \overline{z - 2}$$
$$D z^2 . \overline{z - 1} . \overline{z - 2} - - = * \quad * \quad 3 D z . \overline{z - 1} . \overline{z - 2} + D z . \overline{z - 1} . \overline{z - 2} . \overline{z - 3}$$
$$E z^2 . \overline{z - 1} . \overline{z - 2} . \overline{z - 3} = * \quad * - - - - * \quad * \quad + 4 E z . \overline{z - 1} . \overline{z - 2} . \overline{z - 3} +$$
&c.

Consequently by gathering the homologous terms into one, the series will be reduced to its first form, and will be

$$T z = \begin{Bmatrix} +A \\ +B \end{Bmatrix} z + \begin{Bmatrix} B \\ +2 C \end{Bmatrix} z . \overline{z - 1} + \begin{Bmatrix} C \\ +3 D \end{Bmatrix} z . \overline{z - 1} . \overline{z - 2} + \begin{Bmatrix} D \\ +4 E \end{Bmatrix} z . \overline{z - 1} . \overline{z - 2} . \overline{z - 3} + \&c.$$

namely, when the ὁμολογία, homology, or likeness, of the terms no where depends on the co-efficients A, B, C, D, &c. but altogether upon the indeterminate quantity z, $\begin{Bmatrix} +A \\ +B \end{Bmatrix} z$, the first term in this series, may be compared with $B z$ the second in the other; likewise the second in this with the third in the other, and so in the rest.

Moreover if in the first equation $T = A + B z + C z . \overline{z - 1} + D z . \overline{z - 1} . \overline{z - 2}$ + &c. be wrote the succeeding values of the variable quantities instead of the present, that is, T' for T, and $z + 1$ for z, it will become

$$T' = A + B \overline{z + 1} + C \overline{z + 1} . z + D \overline{z + 1} . z . \overline{z - 1} + E \overline{z + 1} . z . \overline{z - 1} . \overline{z - 2} + \&c.$$

But $A - - - - - - - - = A$
$$B \overline{z + 1} - - - - - = B + B z$$
$$C \overline{z + 1} . z - - - - - = * \quad 2 C z + C z . \overline{z - 1}$$
$$D \overline{z + 1} . z . \overline{z - 1} - - = * - - * - 3 D z . \overline{z - 1} + D z . \overline{z - 1} . \overline{z - 2}$$
$$E \overline{z + 1} . z . \overline{z - 1} . \overline{z - 2} . = * - - * - - * - - - - 4 E z . \overline{z - 1} . \overline{z - 2} + \&c.$$

And from thence

$$T' = \begin{Bmatrix} +A \\ +B \end{Bmatrix} z + \begin{Bmatrix} B \\ +2 C \end{Bmatrix} z + \begin{Bmatrix} C \\ +3 D \end{Bmatrix} z . \overline{z - 1} + \begin{Bmatrix} D \\ +4 E \end{Bmatrix} z . \overline{z - 1} . \overline{z - 2} + \begin{Bmatrix} E \\ +5 F \end{Bmatrix} z . \overline{z - 1} . \overline{z - 1} . \overline{z - 3}$$

+ &c. which is the form desired.

But observe the foundation of these operations; the quantity to be reduced may be reduced by multiplication to the powers of the indeterminate quantity z; and then work according to the following *Example*:

Suppose, for example, $\overline{z - 3} \times z \times z \overline{z + 1} \times \overline{z + 4}$ be the quantity to be reduced; imagine $\overline{z - 3} \times z \times z \overline{z + 1} \times \overline{z + 4} = a z . \overline{z - 1} . \overline{z - 2} . \overline{z - 3} + b z . \overline{z - 1}$.
$\overline{z - 2} + c z . \overline{z - 1} + d z$.

Where the greatest number of factors in the quantity resolv'd is equal to the number of those in the quantities to be resolved, let each quantity be reduced to the powers of the indeterminate quantity, by multiplication, and it will be

$$z^4 +$$

$$z^4 + 2z^3 - 11z^2 - 12z = az^4 \begin{matrix} -6a \\ +b \end{matrix}\Big\} z^3 \begin{matrix} +11a \\ -3b \\ +c \end{matrix}\Big\} z^2 \begin{matrix} -6a \\ +2b \\ -c \\ +d \end{matrix}\Big| z$$

And, by comparing the homologous terms, we shall have $a=1$, $b-6a=2$, $c-3b+11a=-11$, $d-c+2b-6a=-12$; from which a will be found $=1$, $b=8$, $c=2$, $d=-20$; hence the proposed quantity becomes $z^4 + 2z^3 - 11z^2 - 12z = z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + 8z \cdot \overline{z-1} \cdot \overline{z-2} + 2z \cdot \overline{z-1} - 20z$.

And after the same manner we may proceed in other cases; but for the sake of brevity take the following rule. *Divide unity by the terms of this progression continually, $n-1$, $n-2$, $n-3$, $n-4$, &c. that is, divide unity by $n-1$, and that quotient by $n-2$, and the last quote by $n-3$,* and so on. Then dispose regularly in a table, as under, all the quotes thus coming out, rejecting the powers of n, and retaining only the coefficients, which only are of use for this purpose, and you will have this

FIRST TABLE.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

Assume now, for the co-efficients, the numbers in the descending columns, and you will have the following values of the powers:

$z = z$

$z^2 = z + z \cdot \overline{z-1}$,

$z^3 = z + 3z \cdot \overline{z-1} + z \cdot \overline{z-1} \cdot \overline{z-2}$,

$z^4 = z + 7z \cdot \overline{z-1} + 6z \cdot \overline{z-1} \cdot \overline{z-2} + z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3}$,

$z^5 = z + 15 \cdot z \cdot \overline{z-1} + 25z \cdot \overline{z-1} \cdot \overline{z-2} + 10z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} \cdot \overline{z-4}$, &c.

Wherefore this table being once had, every quantity is reduced to the form sought,

fought, without a tedious *computus*. Let this $z^4+2z^3-11z^2-12z$ be propofed to be reduced; take the values of the powers from the table, and draw them refpectively into their co-efficients -12, -11, $+2$, and 1, and you will obtain

$$-12z=-12z,$$
$$-11z^2=-11z-11z.\overline{z-1},$$
$$+2z^3=+2z+6z.\overline{z-1}+2z.\overline{z-1}.\overline{z-2},$$
$$+z^4=+z+7z.\overline{z-1}+6z.\overline{z-1}.\overline{z-2}+z.\overline{z-1}.\overline{z-2}.\overline{z-3},$$
$$z^4+2z^3-11z^2-12z=-20z+2z.\overline{z-1}+8z.\overline{z-1}.\overline{z-2}+z.\overline{z-1}.\overline{z-2}.\overline{z-3}.$$

And the values of the members being gathered into one fum, gives the total value, as already hath come out.

It is here to be obferv'd, that an infinite feries, made of the afcending powers of the indeterminate quantity, cannot be reduced generally into another of the aforefaid form; for every co-efficient would be an infinite feries: but in finite feries, the thing fucceeds, as fhewn above.

Alfo the feries of the other form are in like manner reduced; for fuppofe any quantity fought be

$$T=\frac{A}{z}+\frac{B}{z.\overline{z+1}}+\frac{C}{z.\overline{z+1}.\overline{z+2}}+\frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$$

Then if it be required to find the fucceffive value of T, write $z+1$ for z, and the fucceffive value will come out

$$T'=\frac{A}{z+1}+\frac{B}{z+1.\overline{z+2}}+\frac{C}{z+1.\overline{z+2}.\overline{z+3}}+\frac{D}{z+1.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

and to reduce the feries to the form of the former, I work thus,

$$\frac{A}{z+1}\quad - - - - - =\frac{A}{z}-\frac{A}{z.\overline{z+1}},$$

$$\frac{B}{z+1.\overline{z+2}}\quad \cdots \cdots =*\quad \frac{B}{z.\overline{z+1}}-\frac{2B}{z.\overline{z+1}.\overline{z+2}},$$

$$\frac{C}{z+1.\overline{z+2}.\overline{z+3}}\cdots =*\cdots *\cdots \frac{C}{z.\overline{z+1}.\overline{z+2}}-\frac{3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}},$$

$$\frac{D}{z+1.\overline{z+2}.\overline{z+3}.\overline{z+4}}=*\cdots \cdots \cdots *\cdots \frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$$

and I have $T'=\dfrac{A}{z}+\dfrac{B-A}{z.\overline{z+1}}+\dfrac{C-2B}{z.\overline{z+1}.\overline{z+2}}+\dfrac{D-3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$

where now the denominators are the fame as in the value of T, and for that reafon we may inftitute a comparifon of the terms, as occafion requires;

quires; and such operations are demonstrated thus; put $\frac{1}{z+1} = \frac{1}{z} - \frac{a}{z \cdot z + 1}$, a being the quantity immediately to be found, then by drawing $z \cdot z + 1$ into the denominator, there will come out $z = z + 1 - a$, or by blotting out z on both sides, $0 = 1 - a$, and $a = 1$; wherefore by substituting unity for a, we shall have $\frac{1}{z+1} = \frac{1}{z} - \frac{1}{z \cdot z + 1}$. Likewise suppose $\frac{1}{z+1 \cdot z+2} = \frac{1}{z \cdot z+1} - \frac{a}{z \cdot z+1 \cdot z+2}$; and by drawing it into the denominator, it will be $z = z + 2 - a$, or $a = 2$, and from thence $\frac{1}{z+1 \cdot z+2} = \frac{1}{z \cdot z+1} - \frac{2}{z \cdot z+1 \cdot z+2}$.

Let now $\frac{A}{z+2} + \frac{B}{z+2 \cdot z+3} + \frac{C}{z+2 \cdot z+3 \cdot z+4} + \frac{D}{z+2 \cdot z+3 \cdot z+4 \cdot z+5} +$ &c. be proposed, which we must reduce into another of due form; work as before, and you will find

$$\frac{A}{z+2} \cdots \cdots = \frac{A}{z} - \frac{2A}{z \cdot z+1} + \frac{2A}{z \cdot z+1 \cdot z+2},$$

$$\frac{B}{z+2 \cdot z+3} \cdots = * \cdot \frac{B}{z \cdot z+1} - \frac{4B}{z \cdot z+1 \cdot z+2} + \frac{6B}{z \cdot z+1 \cdot z+2 \cdot z+3}$$

$$\frac{C}{z+2 \cdot z+3 \cdot z+4} \cdots = * \cdots * \cdots \frac{C}{z \cdot z+1 \cdot z+2} - \frac{6C}{z \cdot z+1 \cdot z+2 \cdot z+3} + \text{\&c.}$$

$$\frac{D}{z+2 \cdot z+3 \cdot z+4 \cdot z+5} = * \cdots * \cdots * \cdots \frac{D}{z \cdot z+1 \cdot z+2 \cdot z+3}$$

— &c. and the proposed series will be found under this due form

$$\frac{A}{z} + \frac{B-2A}{z \cdot z+1} + \frac{C-4B+2A}{z \cdot z+1 \cdot z+2} + \frac{D-6C+6B}{z \cdot z+1 \cdot z+2 \cdot z+3} + \text{\&c.}$$ and so in other cases.

If the fraction to be reduc'd be $\frac{1}{z+1}$, two members will be in its value thereof, as in the first example; if it be $\frac{1}{z+2}$, there will be three, as in the last example; and in general in the value of $\frac{1}{z+n}$ reduced to a due form, the number of members will exceed the number n by unity.

Here

Here I suppose n to be a whole and affirmative number; for if it be a broken or negative number, the value of the fraction $\frac{1}{z+n}$ will run out into an infinite series; and the following is the general rule for such like transmutations.

Draw the terms of this progression n, $1+n$, $2+n$, $3+n$, &c. continually into one another, and let the products be disposed as in the following table, according to the order of the powers of n, only preserving the co-efficients, and there will come out this TABLE.

1								
1	1							
2	3	1						
6	11	6	1					
24	50	35	10	1				
120	274	225	85	15	1			
720	1764	1624	735	175	21	1		
5040	13068	13132	6769	1960	322	28	1	
40320	109584	105056	67284	22449	4536	546	36	1
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c. &c.

Then by taking the co-efficients from the descending columns, you will obtain the values of the powers,

$$\frac{1}{z^1} = \frac{1}{z.z+1} + \frac{1}{z.z+1.z+2} + \frac{2}{z.z+1.z+2.z+3} + \frac{6}{z.z+1.z+2.z+3.z+4} + \&c.$$

$$\frac{1}{z^3} = \frac{1}{z.z+1.z+2} + \frac{3}{z.z+1.z+2.z+3} + \frac{11}{z.z+1.z+2.z+3.z+4} + \&c.$$

$$\frac{1}{z^4} = \frac{1}{z.z+1.z+2.z+3} \text{ into } 1 + \frac{6}{z+4} + \frac{35}{z+4.z+5} + \frac{225}{z+4.z+5.z+6} +$$
&c. and so in others.

Therefore having got a series composed of the powers, it may always be reduced into another of a desired form, by the help of this table.

Or let this series $\frac{A}{z^1} + \frac{B}{z^3} + \frac{C}{z^4} + \frac{D}{z^5} +$ &c. be proposed, take the co-efficients out of the transverse columns, and put

$$a = A,$$

$$a = A,$$
$$b = A + B,$$
$$c = 2A + 3B + C,$$
$$d = 6A + 11B + 6C + D,$$
$$e = 24A + 50B + 35C + 10D + E,$$
$$f = 120A + 274B + 225C + 85D + 15E + F, \&c.$$

And the series composed of the powers will be transformed into the following one of due form,

$$\frac{a}{z.z+1} + \frac{b}{z.z+1.z+2} + \frac{c}{z.z+1.z+2.z+3} + \frac{d}{z.z+1.z+2.z+3.z+4} +$$
$$\&c.$$

Let this fraction $\frac{1}{zz+nz}$ be now proposed, first by division I resolve it into this common series $\frac{1}{z^2} - \frac{n}{z^3} + \frac{n^2}{z^4} - \frac{n^3}{z^5} + \frac{n^4}{z^6} - \&c.$ whence $A=1$, $B=-n$, $C=+n^2$, $D=-n^3$, $E=+n^4$ &c. and these values being substituted, there will emerge $a=1$, $b=1-n$, $c=2-3n+n^2$, $d=6-11n+6n^2-n^3$ &c. and consequently $\frac{1}{zz+nz} = \frac{1}{z.z+1} + \frac{1-n}{z.z+1.z+2} + \frac{2-3n+n^2}{z.z+1.z+2.z+3} +$

$\frac{6-11n+6n^2-n^3}{z.z+1.z+2.z+3.z+4} + \&c.$ that is $\frac{1}{zz+nz} = \frac{1}{z.z+1} + \frac{1-n}{z+2}A + \frac{2-n}{z+3}$

$B + \frac{3-n}{z+4}C + \frac{4-n}{z+5}D + \&c.$ where the quantities A, B, C, D &c. denote the terms of this series according to Sir *Isaac Newton*'s method; and it appears that the series terminates as often as n is a whole affirmative number; likewise in other examples, let z denote the *least* factor in the denominator, and the series will always terminate by this method, when the nature of the thing will bear. As suppose this fraction $\frac{1}{x.x-3.x+2}$, I put $z=x-3$, as being the least of the three factors; then it will be $x=z+3$, and $x+2=z+5$: And the fraction will become $\frac{1}{z.z+3.z+5}$, or by multiplication it is $\frac{1}{z^3+8z^2+15z}$, and by division it is $\frac{1}{z^3} - \frac{8}{z^4} + \frac{49}{z^5} - \frac{272}{z^6} + \frac{1441}{z^7} - \frac{7448}{z^8} + \frac{37969}{z^9} - \&c.$ whence $A=0$, $B=1$, $C=-8$, $D=+49$, $E=-272$, $F=+1441$, $G=-7448$, &c. and from thence $a=0$, $b=1$, $c=-5$, $d=12$; $e=-12$, but f and the rest are nothing, and consequently the series breaks off, being accurately

$$\frac{1}{z.z+3.z+5} = \frac{1}{z.z+1.z+2} \text{ into } 1 - \frac{5}{z+3} + \frac{12}{z+3.z+4} - \frac{12}{z+3.z+4.z+5}$$

In any fraction $\dfrac{1}{z.z+a.z+b.z+c. \&c.}$, let z be the leaft of the factors, fo that a, b, c, &c. be affirmative, and if they be alfo whole numbers, the feries will be terminated, otherwife will run out into an infinite feries: But when the feries breaks off, it may be found many ways, more elegantly than by the above general rule, for it is quite abfurd to reduce firft a finite fraction into an infinite feries, to find its value afterwards in finite terms: Which we have here done, becaufe we would illuftrate the general rule, not that it would be the beft method when the feries breaks off.

If in the firft table be taken the numbers out of the afcending columns, and there be put

$$a = A,$$
$$b = B - A,$$
$$c = C - 3B + A,$$
$$d = D - 6C + 7B - A,$$
$$e = E - 10D + 25C - 15B + A,$$
$$f = F - 15E + 65D - 90C + 31B - A, \&c.$$

Then a feries of this form $\dfrac{A}{z.z+1} + \dfrac{B}{z.z+1.z+2} + \dfrac{C}{z.z+1.z+2.z+3}$

$+$ &c. will pafs into this $\dfrac{a}{z} + \dfrac{b}{z^2} + \dfrac{c}{z^3} + \dfrac{d}{z^4} +$ &c. which is compofed of the powers.

In thefe tranfmutations, we have made no comparifon of the term $\dfrac{A}{z}$, becaufe without any tranfmutation it ambiguoufly belongs as well to the feries of powers as to that of the factors.

OF THE

SUMMATION of SERIES.

PART I.

IN this first part I have endeavoured to abbreviate the methods of computation in the quadrature of curves, and likewise in those problems that are more difficult, and that, by coming to the values of infinite series more expeditiously than by a simple addition of terms, as is commonly done; in series converging swiftly, this presently dispatches the business, nor is there need of any further artifice. Nevertheless, when they converge slowly, there is immense labour very often required, and indeed the greater, by how much less is the convergency; thus if they approximate very slowly, they come out intirely intractable, for it is very well known that there is sometimes need of above a thousand terms, to get the sum accurately to two or three figures. We will therefore shew, in the following sheets, an expeditious method of transforming those series converging very slow, into others converging very swift, of which last, the sums may be computed with very little trouble to a great many places of figures.

The series transformed will indeed break off, when the series to be summed are summable; and in this case transmutation will become summation; but concerning summable series, I have been the less sollicitous, and only speak of them by the bye, as they very seldom occur in the quadratures of figures. For I have not here bestowed my labour in finding out useless series, which are summable by theorems well known, but in finding out theorems by which useful series may expeditiously be summed to so many places of figures, as shall be necessary for any one's purpose.

Of Series that are more simple.

Not only the converging of series conduces very much for contracting the methods of calculation, but also its simplicity; wherefore, before we can come to transmutations, we must know that *Newton's* series, in his quadrature of curves, not only breaks off, when the nature of the thing will suffer it, but also that they are the most simple of all, when they go

in

in infinitum, and therefore they are preferable to those series found by the common method, namely, by reducing the ordinates into converging series, that the areas may from thence be computed.

Let $x^{\theta-1} \times \overline{e+fx^n}|^{\lambda-1}$ be the ordinate of a curve, in which x is the abscissa, e and f co-efficients, and $\theta-1$, $\lambda-1$, and n indices of the powers; put $r=\dfrac{\theta+n}{n}$, $s=\dfrac{\theta+\lambda n}{n}$, and, according to *Newton*, the area will be $\dfrac{x^\theta}{\theta e} \times$

$\overline{e+fx^n}|^\lambda - \dfrac{s}{r}A\dfrac{fx^n}{e} - \dfrac{s+1}{r+1}B\dfrac{fx^n}{e} - \dfrac{s+2}{r+2}C\dfrac{fx^n}{e} - \dfrac{s+3}{r+3}D\dfrac{fx^n}{e} -$ &c. where A, B, C, D, &c. denote the terms, every one in its order from the beginning; namely, $A=\dfrac{x^\theta}{\theta e} \times e+fx^n|^\lambda$, $B=-\dfrac{s}{r}A\dfrac{fx^n}{e}$, $C=-\dfrac{s+1}{r+1}B\dfrac{fx^n}{e}$, &c. Let now be proposed the finding of the arc from the right sine x being given, or, which is all one, the quadrature of a curve whose ordinate is $\dfrac{1}{\sqrt{1-xx}}$: this reduced to a due form, becomes $x^\theta \times \overline{1-xx}|^{-\frac{1}{2}}$, which, being compared with the general ordinate, gives $e=1$, $f=-1$, $n=2$, $\theta-1=0$, $\lambda-1=-\frac{1}{2}$; therefore $\theta=1$, $\lambda=\frac{1}{2}$, and from thence $r=\frac{1}{2}$, $s=1$; which being substituted in the theorem, there arises for the arc this series $x\sqrt{1-xx} + \frac{1}{3}Ax^2 + \frac{4}{5}Bx^2 + \frac{6}{7}Cx^2 + \frac{8}{9}Dx^2 + \frac{10}{11}Ex^2 +$ &c.

But if the ordinate proposed be first resolved into a series by *Newton's* theorem for involving a binomial, and taking the fluent of every term, there will come out for the same arc this series $x + \frac{1\cdot1}{2\cdot3}Ax^2 + \frac{3\cdot3}{4\cdot5}Bx^2 + \frac{5\cdot5}{6\cdot7}Cx^2 + \frac{7\cdot7}{8\cdot9}Dx^2 + \frac{9\cdot9}{10\cdot11}Ex^2 +$ &c. whence it appears that the first is by far the more simple, and consequently easier to be continued *in infinitum*. Suppose, for *example*, the arc sought be the eighth part of the whole circumference, its sine x will be $=\sqrt{\frac{1}{2}}$, which being substituted, the series will be

First $\frac{1}{2} + \frac{1}{3}A + \frac{4}{5}B + \frac{6}{7}C + \frac{8}{9}D + \frac{10}{11}E +$ &c.

Second $\sqrt{\frac{1}{2}} + \frac{1}{2\cdot3}A + \frac{3}{4\cdot5}B + \frac{5}{6\cdot7}C + \frac{7}{8\cdot9}D + \frac{9}{10\cdot11}E +$ &c.

In which case the first is preferable for two reasons; as, first, it is produced by more simple factors; and, secondly, because it is free from the surd number which is found in the second; nevertheless when x is a rational quantity, and $x\sqrt{1-xx}$ ir-rational, the second series is to be chosen, so that x be of a magnitude so small as to make the series converge very swift; for by this means the extraction of the square root is avoided. Moreover if $x=1$, we must necessarily recur to the second, because in that case the quantity $\sqrt{1-xx}$ vanishes, into which the first is multiplied.

Of SERIES *converging swifter.*

WHEN an indeterminate quantity, (the quantity increasing which is desired) grows very large, and at last infinitely great, the terms of the series composed therefrom will be alternately negative and affirmative, and will converge flower than when the indeterminate quantity cannot increase beyond a given magnitude. As, for instance, suppose the area of a circular arc be sought, it is better to apply the right sine, which cannot be greater than radius, than the tangent, which soon grows to an immense length, as *Newton* formerly observed; and, on the contrary, tangents are preferable in the hyperbola, because they cannot exceed a given magnitude, but are contained within determined limits, and those sufficiently narrow. But what we have here said does not hinder, but that the arc, or area, of a small magnitude, may be found, as one has a mind, for the difference is only remarkable in those cases in which the quantities sought are great. Those series whose terms are alternately negative and affirmative are more manageable than others in summation; and what hath here been said concerning binomial curves, holds in those of superior nomes.

It is, indeed true, that series converging swiftly may be found many ways, by the help of *Newton*'s differential method; but the more they converge, the more they use to be compounded; wherefore I prefer those that are more simple, tho' they converge flower.

Of SUCCESSIVE SUMS.

BY a successive sum I understand a quantity which succeeds the sum of all the terms, when the subsequent terms take the places of the antecedent. Thus, if the sum be $T+T^I+T^{II}+T^{III}+T^{IV}+T^V+$ &c, write the latter terms for the former, and you will have the successive sum $T^I+T^{II}+T^{III}+T^{IV}+T^V$ &c. in which again, if there be substituted the consequent terms for the antecedent, there will come out the sum $T^{II}+T^{III}+T^{IV}+T^V+T^{VI}+$ &c. which succeeds the last, &c.

Hence if S, S^I, S^{II}, S^{III}, &c. denote the successive sums, then will

$$S = T+T^I+T^{II}+T^{III}+T^{IV}+ \text{&c.}$$
$$S^I = -\ -\ T^I+T^{II}+T^{III}+T^{IV}+ \text{&c.}$$
$$S^{II} = -\ -\ -\ -\ T^{II}+T^{III}+T^{IV}+ \text{&c.}$$
$$S^{III} = -\ -\ -\ -\ -\ -\ T^{III}+T^{IV}+ \text{&c.}$$

That is, if from any infinite series be subducted the first term, and from the remainder be also subducted the first term, and from this remainder which is now left be again subducted the first term, and so on at pleasure,

the

the feries after this manner gradually lofing the firft terms, the fucceffive fums will be as follows.

$$S^I = S - T, \quad S^{II} = S^I - T^I, \quad S^{III} = S^{II} - T^{II}, \text{ \&c.}$$

$$\text{Or} \quad S^I = S - T,$$
$$S^{II} = S - T - T^I,$$
$$S^{III} = S - T - T^I - T^{II},$$
$$S^{IV} = S - T - T^I - T^{II} - T^{III},$$
$$\text{\&c.}$$

I have here fpoke concerning the fums of all the terms *in infinitum*, which begin at any given term; for whatfoever term T is, S will be the fum thereof, and of all the following terms, likewife S¹ will be the fum of T¹, and of all the reft. Thefe things hold when the terms are infinite in number, but when we are confidering the fum of a finite number of terms, S will be the fum of all the terms from the beginning, to any given term T, and S¹ will be the fum of the fame terms when T is taken away, and S¹¹ will be the fum of the fame terms when T, T¹ are taken away, &c.

Hence in the fummation of terms from any given term *ad infinitum*, if z be the length of an abfciffa which correfponds to the fum S, then $z+1$, $z+2$, $z+3$, &c. will be the lengths of the fame refpectively correfponding to the fucceffive fums S¹, S¹¹, S¹¹¹, &c. And, on the contrary, in fummation of terms from any given one to the beginning of the feries, the lengths $z-1$, $z-2$, $z-3$, &c. will anfwer to the fums S¹, S¹¹, S¹¹¹, &c. whilft the abfciffa z anfwers to S; for in the firft cafe the diftances of the fums from the beginning perpetually increafe with the increment of the abfciffa, and they decreafe by the fame decrement in the latter cafe.

Let S S¹ be any curve, whofe afymptote is ab, and abfciffa A B parallel
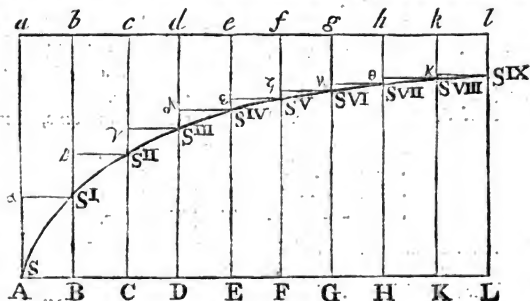
thereto;

thereto; divide the abscissa into an infinite number of equal parts, as AB, BC, CD, &c. and from the points A, B, C, D, &c. erect perpendiculars to the asymptot cutting the curve in the points S, S^I, S^{II}, &c. and the asymptot in a, b, c, &c. From the points S^I, S^{II}, S^{III}, &c. to the next antecedent ordinates, draw $S^I\alpha$, $S^{II}\beta$, $S^{III}\gamma$, $S^{IV}\delta$, &c. parallel to the abscissa; so that $S\alpha$, $S^I\beta$, $S^{II}\gamma$, $S^{III}\delta$, &c. be the differences of the ordinates, as well those which are extended from the curve to the asymptot, as those which are extended from the curve to the abscissa. Therefore the ordinates, intercepted between the curve and the asymptot, will expound the sums; and their differences going from thence *in infinitum* will expound the terms; *i.e.* if eS^{IV} denotes the sum, the succeeding ordinates will be fS^V, gS^{VI}, hS^{VII}, &c. whose differences εS^{IV}, ζS^V, ηS^{VI}, &c. *in infinitum* are the terms whose sum is eS^{VI}. And likewise if ES^{IV}, DS^{III}, CS^{II}, &c. denote the successive sums, whose first is ES^{IV}, the antecedent differences δS^{III}, γS^{II}, &c. will exhibit a finite number of terms, going from the ordinate ES^{IV} to the beginning of the series. Therefore the summation of series is reduced to the finding of the ordinates from their given differences. But it must be noted, that the last sum is nothing in each case; which will always be, when the curve passes thro' the point A in the abscissa, and at the same time hath ab for the asymptot: This caution must be used, that the sums to be investigated, by the methods hereafter delivered, may be true, wanting no correction, as very often happens in the quadrature of curves.

P R O P O S I T I O N　I.

IF the terms of any series be formed by writing the numbers 1, 2, 3, 4, 5, &c. for z, in the quantity $A + Bz + Cz \cdot \overline{z-1} + Dz \cdot \overline{z-1} \cdot \overline{z-2} + Ez \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} +$ &c. then the sum of the terms from the beginning whose number is z, will be $Az + \overline{z+1}$ into $\frac{1}{2}Bz + \frac{1}{3}Cz \cdot \overline{z-1} + \frac{1}{4}Dz \cdot \overline{z-1} \cdot \overline{z-2} + \frac{1}{5}Ez \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} +$ &c.

Where it is to be noted that the quantity $z+1$ is drawn into the whole series that immediately follows it; and the proposition is thus demonstrated: Suppose the sum $S = Az + \overline{z+1}$ into $\frac{1}{2}Bz + \frac{1}{3}Cz \cdot \overline{z-1} + \frac{1}{4}Dz \cdot \overline{z-1} \cdot \overline{z-2} +$ &c. or $S = Az + \frac{1}{2}B\overline{z+1} \cdot z + \frac{1}{3}C\overline{z+1} \cdot z \cdot \overline{z-1} + \frac{1}{4}D\overline{z+1} \cdot z \cdot \overline{z-1} \cdot \overline{z-2} +$ &c. Then write the succeeding values of the variable ones instead of the present; that is, $S - T$ for S, and $z-1$ for z; and you will have $S - T = A\overline{z-1} + \frac{1}{2}Bz \cdot \overline{z-1} \cdot + \frac{1}{3}Cz \cdot \overline{z-1} \cdot \overline{z-2} + \frac{1}{4}Dz \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} +$ &c. then subducting this equation from the former, there will remain $T = A + Bz + Cz \cdot \overline{z-1} + Dz \cdot \overline{z-1} \cdot \overline{z-2}$

F　　　　　　　　　$+$ &c.

+ &c. Whence, on the contrary, if this value of the term be given as in the propofition, the fum will be that which is affigned. Moreover this fum is nothing when z is nothing, and confequently the theorem is manifeft. *Q. E. D.*

E X A M P L E   I.

Let there be given a feries of the natural numbers 1, 2, 3, 4, 5, 6, &c. thefe are formed by writing 1, 2, 3, &c. for z; comparing it with the term in the theorem, it will be A=o, B=1, and C, D, E, and the following will be equal to nothing; which values being fubftituted, there comes out $\overline{z+1}$ into $\frac{1}{2}z$ or $\frac{zz+z}{2}$ for the fum of fo many of the terms of the feries propofed, as there are units in z. Thus if $z=6$, there will come out $\frac{36+6}{2}=21$ for the fum of the firft fix terms.

E X A M P L E   II.

Let there now be given a feries of the odd numbers 1, 3, 5, 7, 9, 11, &c. thefe are formed by writing 1, 2, 3, 4, &c. in the quantity $2z-1$, that is, $-1+2z$, which compared with the general value of the term, gives A=−1, B=2, C, D, E, &c. nothing, which being wrote in the fum, there comes out $-z+\overline{z+1}$ into $\frac{2z}{2}$, or zz for the aggregate of fo many terms as z fignifies. And fo it is in the prefent cafe, for the fucceffive fums are the fquares of the natural numbers.

E X A M P L E   III.

Suppose a feries of fquares, as 1, 4, 9, 16, 25, 36, 49, &c. were to be fummed, which are formed by this expreffion zz. Now zz reduced to the form of the theorem, according to the rules delivered in the introduction, is $z+z.\overline{z-1}$; confequently A=o, B=1, C=1, and the fum from thence becomes $\overline{z+1}$ into $\frac{z}{2}+\frac{z.\overline{z-1}}{3}$; that is, $\frac{z.\overline{z+1}.\overline{2z+1}}{6}$. Suppofe for *example*, 7 be wrote for z, you will then have $\frac{7.8.15}{6}=140$, which is the fum of feven terms.

E X A M P L E   IV.

Let now be propofed the fquares of the odd numbers 1, 9. 25, 49, 81, 121, 169, &c. Thefe are formed by writing 1, 2, 3, 4, &c. fucceffively

cessively in the expression $1 + 4zz - 4z$, which being wrote thus, $1 + 4z . \overline{z-1}$, gives $A=1$, $B=0$, $C=4$, D, E &c. $=0$; and these substituted, the sum comes out $z + \overline{z+1}$ *into* $\frac{1}{3}z . \overline{z-1}$, or $\frac{4z^3-z}{3}$.

E X A M P L E V.

IF there be given the cubes 1, 8, 27, 64, 125, 216, &c. which z^3 assigns, reduce z^3 to its due form, $z + 3z . \overline{z-1} + z . \overline{z-1} . \overline{z-2}$, then $A=0$, $B=1$, $C=3$, $D=1$, and the rest will be nothing; therefore the sum is $\overline{z+1}$ *into* $\frac{1}{2}z + z . \overline{z-1} + \frac{1}{4}z . \overline{z-1} . \overline{z-2}$ or $\frac{zz}{4} \times \overline{z+1}\,|^2$. And hence it appears that the sums of these cubes are the squares of the numbers 1, 3, 6, 10, 15, &c. namely of triangular numbers.

S C H O L I U M.

SUCH series are easier summed by the differences of terms; for let A, A_2, A_3, &c. denote a series to be summed; take the first differences of the terms B, B_2, B_3, &c. the second C, C_2, C_3, &c. the third D, D_2, &c. and so on till you come to the last, which here is E, and the sum of the terms whose

A	A_2	A_3	A_4	A_5
B	B_2	B_3	B_4	
C	C_2	C_3		
D	D_2			
E				

number is z, will be $A\frac{z}{1} + B\frac{z}{1} \times \frac{z-1}{2} + C\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + D\frac{z}{1} \times \frac{z-1}{2} \times$

$\frac{z-2}{3} \times \frac{z-3}{4} +$ &c.

But it is to be observed, that the differences must be taken, by subducting the former from the latter, that is, by putting $B = A_2 - A$, $B_2 = A_3 - A_2$, &c. also $C = B_2 - B$, &c. And this demonstration depends on *Newton*'s differential method.

Let these series be proposed 1, -1, 0, 8, 27, 61, 114, 119, &c. and taking the differences according to the above method, we shall find $A=1$, $B=-2$, $C=3$, $D=4$, and the rest

$$1, -1, \quad 0, \quad 8, \quad 27, \quad 61, \quad 114, \quad 190$$
$$-2, \quad 1, \quad 8, \quad 19, \quad 34, \quad 53, \quad 76$$
$$3, \quad 7, \quad 11, \quad 15, \quad 19, \quad 23$$
$$4, \quad 4, \quad 4, \quad 4,$$

are nothing; consequently the sum will be $\frac{z}{1} - 2 \times \frac{z}{1} \times \frac{z-1}{2} + 3 \times \frac{z}{1} \times \frac{z-1}{2}$

$\times \frac{z-2}{3} + 4 \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4}$, which reduced into order, becomes

$$z-3$$

$$\overline{z-3}.\overline{z-2}.z.\overline{z+2}\over 6.$$ And the series is formed by writing 0, 1, 2, 3, 4,

&c. in the quantity $\dfrac{4z^3-3zz-13z+6}{6}.$

P R O P O S I T I O N II.

IF the terms of any series whatsoever be formed by writing any numbers differing by unity in the quantity, $\dfrac{A}{z.\overline{z+1}}+\dfrac{B}{z.\overline{z+1}.\overline{z+2}}+$

$\dfrac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\dfrac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+$ &c. the sum of all the

terms in infinitum, beginning at any given term, will be $\dfrac{A}{z}+\dfrac{B}{2z.\overline{z+1}}+$

$\dfrac{C}{3z.\overline{z+1}.\overline{z+2}}+\dfrac{D}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+$ &c.

Put the sum $S=\dfrac{A}{z}+\dfrac{B}{2z.\overline{z+1}}+\dfrac{C}{3z.\overline{z+1}.\overline{z+2}}+\dfrac{D}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+$&c.
Then write the values of S and z, the consequent for the antecedent, that is, $S-T$ for S, and $z+1$ for z, because we are treating of terms infinite in number, and there will come out $S-T=\dfrac{A}{\overline{z+1}}+\dfrac{B}{2.\overline{z+1}.\overline{z+2}}+$

$\dfrac{C}{3.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\dfrac{D}{4.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+$ &c. This equation, subducted from the former, leaves $T=\dfrac{A}{z.\overline{z+1}}+\dfrac{B}{z.\overline{z+1}.\overline{z+2}}+$

$\dfrac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\dfrac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+$ &c. Whence, on the contrary, if this term be given, the sum will be that which is assigned in the proposition. *Q. E. D.*

C O R O L L A R Y I.

IF the term be $\dfrac{p}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4},}$ &c. reject the last factor,
then divide the remainder by the number of factors which remain, and you will have the sum of the terms. Suppose the term be $\dfrac{A}{z.\overline{z+1}}$; re-
ject the last factor $z+1$, and there will remain $\dfrac{A}{z}$, and when there is only

one remaining factor z, then $\frac{A}{z}$ will be the sum of all the terms.

Suppose the term be constituted of three factors, as $\frac{B}{z.z+1.z+2}$; reject the last factor $z+2$, and there will remain $\frac{B}{z.z+1}$, which divided by 2, the number of factors remaining, exhibits $\frac{B}{2z.z+1}$ for the sum.

Likewise, if from the term $\frac{C}{z.z+1.z+2.z+3}$, composed of 4 factors, the last $(z+3)$ be rejected, and the remainder divided by 3, you will have for the sum, $\frac{C}{3z.z+1.z+2}$.

Suppose the term be $\frac{A}{z}$, reject the factor z, and because nothing remains, divide A by o, and you will have for the sum a quantity infinitely great, of which Dr *Brook Taylor* was the first, that I know of, that handled this in his *Methodus Incrementorum* ; likewise M. *Nichol* hath largely and elegantly treated of this in the *Memoires de l' Academie Royale de Sciences*.

C O R O L L A R Y II.

By what we have said of this matter in the Introduction, it appears that any term $\frac{A}{z.z+a.z+b.z+c,\&c.}$ may always be resolved into two summable terms, or perhaps more, and finite in number, when a, b, c, &c. are whole numbers ; therefore, in this case, the series will be summable. As suppose the term be $\frac{1}{z.z+3}$, it is resolved into three summable terms $\frac{1}{z.z+1} - \frac{2}{z.z+1.z+2} + \frac{2}{z.z+1.z+2.z+3}$; whence, by the preceding *Corollary*, the sum will be $\frac{1}{z} - \frac{1}{z.z+1} + \frac{2}{3z.z+1.z+2}$, which joined together are $\frac{3z^2+6z+2}{3z^3+9z^2+6z}$. And likewise if the term be of this form, $\frac{a+bz^1+cz^1+dz^4+\&c.}{z.z+a.z+b.z+c.z+d, \&c.}$, the series will be summable ; so that a, b, c, d, &c. be whole numbers ; and the number of factors in the denominator exceeds the highest dimension of z in the numerator, at least by 2. But I except those cases in which two or more factors in the denominator

G

ᴅominator are equal to one another; and then they are not ſummable.

E X A M P L E   I.

Lᴇᴛ this ſeries $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \frac{1}{10.13.16} + \frac{1}{13.16.19} +$ &c. be propoſed to be ſummed. The terms of this ſeries are aſſigned by the quantity $\frac{1}{3z \cdot 3z+3 \cdot 3z+6}$, as will appear by writing $\frac{1}{3}$, $1\frac{1}{3}$, $2\frac{1}{3}$, $3\frac{1}{3}$, &c. ſucceſſively for z, that is, by this $\frac{1}{27 \cdot z \cdot z+1 \cdot z+2}$; whence the ſum is $\frac{1}{54z \cdot z+1}$; now if in this laſt there be wrote $\frac{1}{3}$, the firſt value for z, there will come out $\frac{1}{24}$ for the ſum of the whole ſeries; and if $1\frac{1}{3}$, its ſecond value, be wrote for z, there will come out $\frac{1}{144}$ for the ſum of the whole ſeries after taking away the firſt term; if for z be wrote $2\frac{1}{3}$, its third value, there will come out $\frac{1}{420}$ for the ſum of the whole ſeries after the two firſt terms are taken away; and ſo *in infinitum*.

E X A M P L E   II.

Lᴇᴛ this ſeries $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7} + \frac{1}{5.8} +$ &c. be propoſed. The terms of this ſeries are aſſigned by $\frac{1}{z \cdot z+3}$, in which 1, 2, 3, 4, &c. muſt be wrote ſucceſſively for z; and $\frac{1}{z \cdot z+3}$ is reduced into three ſummable terms, namely, $\frac{1}{z \cdot z+1} - \frac{2}{z \cdot z+1 \cdot z+2} + \frac{2}{z \cdot z+1 \cdot z+2 \cdot z+3}$; whence the ſum will be $\frac{1}{z} - \frac{1}{z \cdot z+1} + \frac{2}{3z \cdot z+1 \cdot z+2}$, or $\frac{3z^2 + 6z + 2}{3z \cdot z+1 \cdot z+2}$. Now to find the ſum of all the terms, ſubſtitute unity for z, and you will obtain $\frac{3+6+2}{3.1.2.3} = \frac{11}{18}$, the value of the ſeries propoſed.

E X A M P L E   III.

Lᴇᴛ there be given this ſeries $\frac{1}{2.3.4.5} + \frac{4}{3.4.5.6} + \frac{9}{4.5.6.7} + \frac{16}{5.6.7.8} +$ $\frac{25}{6.7.8.9} +$ &c. where the numerators are the ſquares of the natural numbers; every term, in general, will be aſſigned by this expreſſion $zz - 2z + 1$

$\dfrac{zz-2z+1}{z.z+1.z+2.z+3}$, 2, 3, 4, 5, &c. being the succeffive values of the in-determinate quantity z; and it is refolved into three fummable terms,

namely, $\dfrac{1}{z.z+1}-\dfrac{7}{z.z+1.z+2}+\dfrac{16}{z.z+1.z+2.z+3}$: whence the fum

is $\dfrac{1}{z}-\dfrac{7}{2z.z+1}+\dfrac{16}{3z.z+1.z+2}$, that is, $\dfrac{6z^2-3z+2}{6.z.z+1.z+2}$, in which if you fubftitute 2 for z, you will have $\frac{1}{16}$ for the value of the feries.

E X A M P L E IV.

LET the value of this feries $\dfrac{1}{1.2.3.4.5}+\dfrac{27}{2.3.4.5.6}+\dfrac{125}{3.4.5.6.7}+\dfrac{343}{4.5.6.7.8}$
+ &c. be required, where the numerators are the cubes of the odd numbers 1, 3, 5, 7, &c. Now 1, 2, 3, 4, 5, &c. being put for the fuc-ceffive values of z, the terms will be affigned by this expreffion $\dfrac{8z^3-12z^2+6z-1}{z.z+1.z+2.z+3.z+4}$, which is refolved into $\dfrac{8}{z.z+1}-\dfrac{84}{z.z+1.z+2}$
$+\dfrac{386}{z.z+1.z+2.z+3}-\dfrac{729}{z.z+1.z+2.z+3.z+4}$.

Therefore the fum is $\dfrac{8}{z}-\dfrac{84}{2z.z+1}+\dfrac{386}{3z.z+1.z+2}-\dfrac{729}{4z.z+1.z+2.z+3}$

that is, $\dfrac{96z^3+72z^2+80z-3}{12z.z+1.z+2.z+3}$; in which if you write the firft value of z, that is, unity for z, you will find $\frac{245}{144}$ to be the value of the feries.

E X A M P L E V.

LET this feries $\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+\dfrac{1}{7.8}+\dfrac{1}{9.10}+$ &c. be propofed, which was found, by my lord *Brounker*, for the quadrature of the hyperbola; every term in general is affigned by this expreffion $\dfrac{1}{4z.z+\frac{1}{2}}$, where the values of z are $\frac{1}{4}$, $1\frac{1}{4}$, $2\frac{1}{4}$, $3\frac{1}{4}$, &c. which being reduced to a fumma-ble form, is $\dfrac{1}{4z.z+1}+\dfrac{1}{8z.z+1.z+2}+\dfrac{1.3}{16z.z+1.z+2.z+3}+$
$\dfrac{1.3.5}{32z.z+1.z+2.z+3.z+4}+$ &c. But becaufe the difference of the Factors, in the expreffion affigning the terms, is a broken number, it runs into an infinite feries; which fhews that the feries is not fummable;
but

but by going back from the term to the fum, it will be $\frac{1}{4z}+\frac{1}{16z \cdot z+1}$ $+\frac{1 \cdot 3}{48z \cdot z+1 \cdot z+2}+\frac{1 \cdot 3 \cdot 5}{128z \cdot z+1 \cdot z+2 \cdot z+3}+$ &c. which is a feries converging fo much the fwifter, by how much greater the quantity z is. But, for an eafier *computus*, put $A=\frac{1}{4z}$, $B=\frac{A}{2z+2}$, $C=\frac{3B}{2z+4}$, $D=\frac{5C}{2z+6}$, $E=\frac{7D}{2z+8}$, $F=\frac{9E}{2z+10}$, &c. and the fum will be $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\frac{1}{5}E+$ &c. In which if for z be fubftituted $\frac{1}{2}$, its firft value, we fhall obtain the value of the whole feries to be fummed; if for z be fubftituted its fecond value, there will come out the fum of all the terms after the firft is taken away; if for z be fubftituted its third value, there will come out the fum of all the terms except the two firft, &c. I therefore fubftitute for z 13 $\frac{1}{2}$ its 14th value, that z may be big enough to make the feries converge fwift, and I obtain $A=\frac{1}{54}$, $B=\frac{1}{29}A$, $C=\frac{3}{31}B$, $D=\frac{5}{33}C$, $E=\frac{7}{35}D$, $F=\frac{9}{37}E$, &c. in which cafe, the fum $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. will be equal to $\frac{1}{27.28}+\frac{1}{29.30}+\frac{1}{31.32}+$ &c. the whole feries to be fummed, when the thirteen firft terms are taken away; I therefore by addition find their fum to be .674285961. And to get the fum of the reft, I find A, B, C, D, &c. by a *calculus* to fo many places of decimals as I have a mind, and when they are found, I divide them refpectively by 1, 2, 3, 4, 5, &c. as under

A=.018518519	.018518519
B= - - 638570	319285
C= - - - 61797	20599
D= - - - 9363	2341
E= - - - - 1873	375
F= - - - - 455	76
G= - - - - 128	18
H= - - - - - 41	5
I= - - - - - - 14	1
	.018861219

And thereby I obtain .018861219 for the fum of all the terms after the thirteenth. Laftly, this added to the aggregate of the initial terms before found, makes .693147180, for the value of the feries to be fummed, that is, for the hyperbolic logarithm of 2.

The more terms are gathered from the beginning of the feries, the
fwifter

fwifter will the feries converge which gives the fum of all the reft, be-caufe of z being fo much the greater. And the excellency of this method appears chiefly in adding the terms to the aggregate of the initial ones, that z may be augmented by fo many unities, whereby the feries tranf-formed will converge *ad libitum*.

But that it is impoffible in practice to obtain the fums of thefe feries by a mere collection of terms, will be manifeft from the following computa-tion, where you have the fum of a hundred, thoufand, ten thoufand, and fo on to ten millions, hundred millions of terms.

$$
\text{The fum of}
\left\{
\begin{array}{l}
100 \\
1000 \\
10000 \\
100000 \\
1000000 \\
10000000 \\
100000000 \\
1000000000
\end{array}
\right\}
\text{Terms}
\left\{
\begin{array}{l}
.690653446 \\
.692897242 \\
.693122181 \\
.693144680 \\
.693146930 \\
.693147155 \\
.693147178 \\
.693147180
\end{array}
\right.
$$

From this calculus it appears that a hundred terms gives the fum accurate to two figures; and if we gradually fum up ten times the number of terms, it is an uncertainty whether another figure can be gain'd: there-fore if any one would find an accurate value of this feries to nine places of figures (which requires no art but only addition) they would require one thoufand million of terms; and this feries converges much fwifter than many others, whofe values are finite quantities.

E X A M P L E VI.

SUPPOSE this feries $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} +$ &c. is to be fummed, where the denominators are the fquares of the natural numbers 1, 2, 3, &c. and any term in general is $\frac{1}{zz}$; this reduced into a fummable form,

is $\frac{1}{z.z+1} + \frac{1}{z.z+1.z+2} + \frac{1.2}{z.z+1.z+2.z+3} + \frac{1.2.3}{z.z+1.z+2.z+3.z+4}$

$+$ &c. Therefore the fum is $= \frac{1}{z} + \frac{1}{2z.z+1} + \frac{1}{3z.z+1.z+2} +$

$\frac{1.2.3}{4z.z+1.z+2.z+3} +$ &c, which, by putting $A = \frac{1}{z}$, $B = \frac{A}{z+1}$, $C =$

$\frac{2B}{z+2}$, $D = \frac{3C}{z+3}$, $E = \frac{4D}{z+4} +$ &c. is $A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D +$ &c. Now

H if

if you substitute 13 its *thirteenth* value for z, you will have the sum of all the terms in the series to be summed after the *twelfth*; in which case A will be $= \frac{1}{17}$, B$= \frac{1}{15}$A, C$= \frac{1}{17}$B, D$= \frac{1}{16}$C, E$= \frac{1}{17}$D, &c. And A$+\frac{1}{3}$B$+\frac{1}{5}$C$+\frac{1}{7}$D, &c. will be the sum of the terms $\frac{1}{169}+\frac{1}{196}+\frac{1}{211}+ \frac{1}{216}+$ &c. the computation of which is as follows.

A=.076923077	.076923077
B= - 5494505	2747252
C= -- 732601	244200
D= --- 137363	34341
E= --- 32321	6464
F= --- 8978	1496
G= ---- 2835	405
H= ---- 992	124
I= ---- 378	42
K= ----- 155	16
L= ----- 67	6
M= ----- 31	3
N= ---- 15	1
	.079957427

Whence we have .079957427 for the sum of the terms $\frac{1}{169}+\frac{1}{196}+\frac{1}{211}$ + &c. which added to the aggregate of the first *twelve*, or 1.564976638, makes 1.644934065 for the value of the whole series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ &c. But this series does not converge so fast as that of my lord *Brounker*, in the above example.

P R O P O S I T I O N   III.

IF the terms of any series be formed by writing any numbers differing by unity, for z in the quantity x^{z+n} into $\dfrac{a}{z}+\dfrac{b}{z.z+1}+\dfrac{c}{z.z+1.z+2}+$

$\dfrac{d}{z.z+1.z+2.z+3}+$ &c. the sum will be equal to x^{z+n} into $\dfrac{a}{1-x.z}+$

$\dfrac{b-Ax}{1-x.z.z+1}+\dfrac{c-2Bx}{1-x.z.z+1.z+2}+\dfrac{d-3Cx}{1-x.z.z+1.z+2.z+3}$, &c.

The quantities A, B, C, D, &c. denote the coefficients of the terms preceding those in which they are found; namely, A$=\dfrac{a}{1-x}$, B$=\dfrac{b-Ax}{1-x}$,

C$=\dfrac{c-2Bx}{1-x}$, &c. But I except that case in which x is equal to unity;

when this happens, the series will be summed by the above proposition ; see the demonstration.

Suppose the sum $S = x^{z+n}$ into $\dfrac{A}{z} + \dfrac{B}{z \cdot z+1} + \dfrac{C}{z \cdot z+1 \cdot z+2} +$

$\dfrac{D}{z \cdot z+1 \cdot z+2 \cdot z+3} + $ &c. then write the values of the variable quantities succeeding S—T, and $z+1$ for the antecedent S and z respectively, and you will have $S-T = x^{z+n+1}$ into $\dfrac{A}{z+1} + \dfrac{B}{z+1 \cdot z+2} + \dfrac{C}{z+1 \cdot z+2 \cdot z+3}$

$+ \dfrac{D}{z+1 \cdot z+2 \cdot z+3 \cdot z+4} + $ &c. that is, $S = x^{z+n}$ into $\dfrac{Ax}{z+1} + \dfrac{Bx}{z+1 \cdot z+2}$

$+ \dfrac{Cx}{z+1 \cdot z+2 \cdot z+3} + \dfrac{Dx}{z+1 \cdot z+2 \cdot z+3 \cdot z+4} + $ &c. which reduced to the form of S, is $S-T = x^{z+n}$ into $\dfrac{Ax}{z} + \dfrac{Bx-Ax}{z \cdot z+1} + \dfrac{Cx-2Bx}{z \cdot z+1 \cdot z+2} +$

$\dfrac{Dx-3Cx}{z \cdot z+1 \cdot z+2 \cdot z+3} + $&c.

Now subtract the value of S—T from the value of S, and there will remain the term $T = x^{z+n}$ into $\dfrac{A \overline{1-x}}{z} + \dfrac{B \overline{1-x} + Ax}{z \cdot z+1} + \dfrac{C \overline{1-x} + 2Bx}{z \cdot z+1 \cdot z+2} +$

$\dfrac{D \overline{1-x} + 3Cx}{z \cdot z+1 \cdot z+2 \cdot z+3} + $ &c.

Lastly, this value of T compared with that in the proposition, gives $A \overline{1-x} = a$, $B \overline{1-x} + Ax = b$, $C \overline{1-x} + 2Bx = c$, &c. which equations exhibit the values of the coefficients as above ; wherefore the value of the sum is rightly assign'd. Q. E. D.

E X A M P L E I.

SUPPOSE this series $1 + \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^3 + \frac{1}{5}t^4 + $ &c. were given to be summed ; the equation to the same is $T = t^{z-\frac{1}{2}}$ into $\dfrac{\frac{1}{2}}{z}$; for by writing $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c. successively for z, there will come out the terms of the series. But by comparing this term with that in the theorem, it will be $x = t$, $n = -\frac{1}{2}$, $a = \frac{1}{2}$; but b, c, d, e, and the other coefficients are $= 0$; and, lastly, these values being wrote, there arises $S = t^{z-\frac{1}{2}}$ into $\dfrac{\frac{1}{2}}{1 - t \cdot z} +$

$\dfrac{At}{1 - t \cdot z \cdot z+1} + \dfrac{2Bt}{t - 1 \cdot z \cdot z+1 \cdot z+2} + \dfrac{3Ct}{t - 1 \cdot z \cdot z+1 \cdot z+2 \cdot z+3} + $ &c.

For *example*, let us put $t = -1$, and the series to be summed will be

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \&c.$$

and the sum will be $S = \pm 1$ into $\dfrac{1}{4z} + \dfrac{A}{2z.z+1} +$

$$\dfrac{2B}{2z.z+1.z+2} + \dfrac{3C}{2z.z+1.z+2.z+3} + \&c.$$

or $S = \pm 1$ into $\dfrac{1}{4z} + \dfrac{A}{2z+2} +$

$$\dfrac{2B}{2z+4} + \dfrac{3C}{2z+6} + \dfrac{4D}{2z+8} + \&c.$$

Where A, B, C, D, &c. denote now all the terms according to the *Newtonian* method, and not the coefficients. And unity with an ambiguous sign, into which the series is multiplied, will be affirmative when $z - \tfrac{1}{4}$ is an even number, but negative when odd. Now gather 12 of the first terms, or, which is all one, fix in this series $\dfrac{2}{1.3} + \dfrac{2}{5.7} + \dfrac{2}{9.11} + \&c.$ two and two in the former series being put into one, the sum will be $=$ .7646006915: then for z write $12\tfrac{1}{2}$, its thirteenth value, and you will have $S = \tfrac{1}{50} + \tfrac{1}{17}A + \tfrac{1}{33}B + \tfrac{1}{77}C + \tfrac{1}{77}D + \tfrac{1}{75}E + \&c.$ which

is a simple series, and converges fast: for ten terms	.0200000000
give $S = .0207974719$, as is evident from the annext	7407407
computation, and if we add the same to the aggre-	510856
gate of the first terms, we shall have .7853981634	49438
for the value of the series to be summed, which we	5992
could never come to by the addition of terms; and	856
by collecting more initial terms, the value of S will	139
approximate much swifter. Therefore by the help	25
of this proposition, the circumference of a circle may	5
with ease be produced to very many figures, by this	1
series converging so very flow, which M. *Leibnitz*	$S = .0207974719$

long ago greatly defired.

The periphery of a circle will be obtain'd very accurately too by the following series of *Newton's*, *viz.* $1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} + \&c.$ where every two terms are alternately negative and affirmative. The same is also effected by this series $1 + \tfrac{1}{5} - \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{11} + \&c.$ in which the denominators conftitute the progreffion of the natural numbers, every third being taken away. The first is equal to a fourth, and the last to a third part of the whole circumference, on fuppofition that the chords of thefe arcs are unities. But before we fhall handle them by this propofition, both muft be divided into two parts, the first into thefe

$$1 - \tfrac{1}{5} + \tfrac{1}{9} - \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{21} + \&c.$$
$$\tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{15} + \tfrac{1}{19} - \tfrac{1}{23} + \&c.$$

The laſt into theſe

$$1 - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{10} + \tfrac{1}{13} - \tfrac{1}{16} + \&c.$$
$$\tfrac{1}{2} - \tfrac{1}{5} + \tfrac{1}{8} - \tfrac{1}{11} + \tfrac{1}{14} - \tfrac{1}{17} + \&c.$$

Then every one of theſe four is to be conſidered apart, and the operation to be managed as in the above example.

E X A M P L E II.

Suppoſe this ſeries $\dfrac{x}{1.2} + \dfrac{x^2}{3.4} + \dfrac{x^3}{5.6} + \dfrac{x^4}{7.8} + \dfrac{x^5}{9.10} + \&c.$ the equation will be $T = x^z$ into $\dfrac{1}{4z \cdot z - \frac{1}{2}}$, which reſolved into a ſeries becomes,

$$T = x^z \text{ into } \frac{1}{4z \cdot z + 1} + \frac{3}{8z \cdot z + 1 \cdot z + 2} + \frac{15}{16z \cdot z + 1 \cdot z + 2 \cdot z + 3} +$$

$$\frac{105}{32z \cdot z + 1 \cdot z + 2 \cdot z + 3 \cdot z + 4} + \&c. \text{ whence by comparing the terms,}$$

it will be $n = 0$, $a = \frac{1}{4}$, $b = \frac{3}{8}$, $c = \frac{15}{16}$, $d = \frac{105}{32}$, $e = \frac{945}{64}$, &c. conſequently therefore $S = x^z$ into $\dfrac{1}{4.1 - x \cdot z} + \dfrac{3 - 8Ax}{8.1 - x \cdot z \cdot z + 1} + \dfrac{15 - 32Bx}{16.1 - x \cdot z \cdot z + 1 \cdot z + 2}$

$+ \dfrac{105 - 96Cx}{32.1 - x \cdot z \cdot z + 1 \cdot z + 2 \cdot z + 3} + \&c.$ the progreſſion of which ſeries is evident to any one. And when the value of x is given in any particular caſe, the ſum will be had as accurate as you pleaſe, *viz.* by firſt adding a ſufficient number of initial terms, to the end that z may be big enough to make the value of S converge faſt. Theſe things being premiſed concerning ſeries whoſe terms are aſſignable, we ſhall now make a tranſition to thoſe which are determined by the relation of the terms.

P R O P O S I T I O N IV.

GIVEN *the relation between the ſucceſſive ſums, to find that which is between the terms.*

In an equation defining the relation between the ſums, ſubſtitute for S^I, S^{II}, S^{III}, &c. their own proper values $S - T$, $S - T - T^I$, $S - T - T^I - T^{II}$, &c. and ſo you will have the equation involving only one ſum S; in which write the conſequent values of the variable quantities for the antecedents, and you will have a new equation involving that ſum S; laſtly, by the help of theſe equations, let S be exterminated, and that which reſults will ſhew the relation of the terms. *Q. E. I.*

I

EXAMPLE

E X A M P L E I.

Suppose an equation to the fums be $\overline{z-n}S=\overline{z-1}S^I$;

For S^I, fubftitute its value $S-T$, and the equation will then be $\overline{n-1}S=\overline{z-1}T$, in which write the fucceeding values of the variable ones for the preceeding ones, that is, $S-T$ for S, T^I for T, and $z+1$ for z; and we fhall have $\overline{n-1}S=\overline{n-1}T+zT^I$, which fubtracted from the former $\overline{n-1}.S=\overline{z-1}.T$, there will remain $\overline{z-n}T=zT^I$; which is the equation for the terms of the feries.

E X A M P L E II.

Let an equation be propofed for the fums $S\times\overline{8zz+20z+9}+3S^I\times\overline{8zz+4z-3}$; fubftitute $S-T$ for S^I, and you will find $S=\frac{1}{3}T\times\frac{8zz+4z-3}{zz+z}$; then, according to the differential method, write $S-T$ for S, T^I for T, and $z+1$ for z, and there will come out $S-T=\frac{3}{3}T^I\times\frac{8zz+20z+9}{zz+3z+2}$; by the help of thefe equations exterminate S, and you will have $\overline{z+2}T+3T^Iz=0$; which is the equation exhibiting the relation of the terms.

And by the fame method are three or more fucceffive fums exterminated.

P R O P O S I T I O N V.

TO *find as many fummable feries as you will.*

The equation for the fums will give the fum of the terms, and that for the terms will give the feries; the firft is affumed at pleafure, and the other is deduced from it by the above propofition; therefore the terms will be had, and alfo their fum. *Q. E. I.*

E X A M P L E I.

Suppose an equation for the fums be $\overline{z-n}S=\overline{z-1}S^I$, as in the firft example of the preceding propofition; you will find that for the terms to be $\overline{z-n}T=zT^I$; and by fubftituting $S-T$ for S^I, the equation for the fums will be $S=\frac{z-1}{n-1}T$. Now let A, B, C, D, &c. denote the terms of this feries, and write in the equation (to the fame) m, $m+1$, $m+2$, $m+3$, &c. fucceffively for z, m being any whole number or fraction, negative or affirmative,

affirmative, and the relation of the terms will be $B = \frac{m-n}{m}A$, $C = \frac{m-n+1}{m+1}B$, $D = \frac{m-n+2}{m+2}C$, $E = \frac{m-n+3}{m+3}D$, &c. Then in the equation $S = \frac{z-1}{n-1}T$, write the firſt term of the ſeries, that is, A for T, and the firſt value of z, that is, m for z; and you will find $S = \frac{m-1}{n-1}A = A + \frac{m-n}{m}A + \frac{m-n+1}{m+1}B + \frac{m-n+2}{m+2}C +$ &c. where we may ſubſtitute any numbers for m and n.

Suppoſe for *example* $m = 5$, $n = 2$, $A = \frac{1}{12}$, then there will come out

$$\frac{1}{3} = \frac{1}{12} + \frac{3}{4}A + \frac{4}{5}B + \frac{5}{6}C + \frac{6}{7}D +$$ &c.

that is, $\frac{1}{3} = \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} +$ &c.

where it is manifeſt the terms are aſſignable; which will always be when n is a whole number: And there will be as many factors in the denominators, as there are units in n. Thus, in the preſent *example*, n is $= 2$, and for that reaſon there are two factors in the denominators of the terms.

Suppoſe now $m = 2$, $n = \frac{3}{2}$, $A = 1$; then there will come out

$$S = 2 = 1 + \frac{1}{4}A + \frac{3}{8}B + \frac{5}{16}C + \frac{7}{0}D + \frac{9}{12}E +$$ &c.

that is, $S = 2 = 1 + \frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + \frac{21}{112} +$ &c.

It is evident now the terms are not aſſignable, becauſe n is a fraction; where it muſt be obſerved, the ſeries terminates, and becomes finite, as often as $m - n$ is *nothing*, or an integer and negative. And if $n - 1$ be nothing, or a negative number, the value of the ſeries will be infinitely great; as appears from the value of the ſum, namely $\frac{m-1}{n-1}A$.

E X A M P L E II.

LET an equation for the ſums be $S \times \overline{8zz + 20z + 9} + 3S^1 \times \overline{8zz + 4z - 3}$, as in the laſt example of the above propoſition, where there was found $S = \frac{1}{12}T \times \frac{8zz + 4z - 3}{zz + z}$, and the relation of the terms was $\overline{z + 2}T + 3zT^1 = 0$; if unity be put for the firſt term, and likewiſe for the firſt value of z, we ſhall obtain $S = \frac{27}{64} = 1 - \frac{3}{4} + \frac{6}{9} - \frac{10}{27} + \frac{15}{81} - \frac{21}{243} +$ &c. where the denominators are the powers of 3, and the numerators triangular numbers.

In theſe examples I have not ſtaid to deduce the ſeries from equations
which

which define the relation of terms, becaufe I fuppofe this is already known from the introduction.

S C H O L I U M.

SUMMATION of feries in the differential method anfwers to quadrature of curves in the method of fluxions, and therefore in both there ufe to arife the like difficulties, which are here to be explained. We have faid that the feries may on every fide be continued *in infinitum* : Suppofe, for *example*, this feries $1+x+x^2+x^3+$ &c. and continued backwards is $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+$ &c. Thefe two feries joined together make of each part only one running *in infinitum*, viz.

$$\&c. \ +\frac{1}{x^4}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2+x^3+x^4 \ \&c.$$

For thefe terms are in continued geometrical progreffion, the antecedents to the confequents as unity to x. In the fummation of this feries we fhall find $\frac{1}{1-x}=1+x+x^2+x^3+$ &c. which indeed is true when x is lefs than unity ; but if it be greater, this feries will be infinitely great, and the fum, $\frac{1}{1-x}$, will not any more be the fum of thefe terms ; but by changing the fign will be equal to the feries proceeding the contrary way, that is, $\frac{-1}{1-x}$, or $\frac{1}{x-1}$, will be $=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+$ &c. and if x be unity, the fum will be $=\frac{1}{1-1}$, and thereby each part of the feries will be infinitely great, inafmuch as it is equal to unity taken an infinite number of times.

After the fame manner if the equation to the feries be $z-nT=zT^1$, and the feries be continued from each part *in infinitum*, one part will converge and the other diverge, except when $n=1$; and the fum $\frac{z-1}{n-1}T$ will always be equal to the converging part of the feries.

So in quadratures, if z^{-n} be an ordinate of an hyperbolic curve, the fluent $\frac{1}{1-n}z^{1-n}$ will exprefs part of the area, lying on this or the other fide of the ordinate according as n is lefs or greater than unity ; but when n is $=1$, the area from each part of the ordinate will be infinitely great, as in the *Apollonian Hyperbola.*

But feries, (although the quantities fought thereby be of a finite magnitude)

nitude) yet they do very often by diverging come out infinitely great, and in these cases continued to opposite parts they sometimes converge, and are equal to the roots required, or differ from them by a determinate quantity. Also they sometimes diverge though continued both ways; but very often they cannot run *in infinitum* both ways, because of impossible terms, or terms infinitely small.

Moreover, as the areas of curves are sometimes to be augmented, and sometimes diminished, by given quantities, to have them true; so likewise the sums found by this proposition sometimes differ from the true ones, in which case they must be corrected by addition or subtraction of a given quantity; namely, when the equation for the sums is such as makes the last of them to be a quantity of a finite magnitude, or infinitely great, there is always need of correction. I will therefore shew, in the following proposition, how an equation must be assumed which shall always make the last sum to be nothing; and by that means the sum found will be the true one, neither by augmenting nor diminishing as has hitherto been shewn.

P R O P O S I T I O N VI.

IF *an equation to the sums be* $S \times \overline{z^\theta + az^{b-1} + bz^{b-1} + \&c.}$ $mS^I \times \overline{z^\theta + cz^{b-1} + dz^{b-2} + \&c.}$ *the last of the sums will be of a finite magnitude in that case only, when* $m = 1$, *and* $a = c$.

For a demonstration of this proposition, we must know that the sum S may be investigated from the equation defining the relation between it and its successive value S^I, nearly in the same manner as you do a fluxional quantity from its equation. For that end we must assume for S a series of this form

$$\frac{z^n}{p^z} \text{ into } A + \frac{B}{z} + \frac{C}{z^1} + \frac{D}{z^1} + \&c.$$

where n, p, A, B, C, D, &c. are invariable quantities. But in the present case, when the sum required is the last of all, consequently at an infinite distance, z will be likewise infinitely great, as being either equal to that distance, or differing from it by a finite quantity; for this reason the latter terms of the series are infinitely less than the former. Therefore, to abbreviate the computation, I reject all the terms after the first, as being useless in this demonstration: So I have $S = \frac{Az^n}{p}$, in which, by writing S^I for S, and $\overline{z+1}$ for z, I obtain $S^I = \frac{A\overline{z+1}^n}{p^{z+1}}$; these values substitute for

K

S and

S and S^I in the equation to the sums, or, which is the same thing, in this equation $S \times \overline{z+a} = mS \times \overline{z+c}$, the remaining terms being omitted for the reasons above given; and there will result $\frac{A z^n}{p^z} \times \overline{z+a} = \frac{m A \overline{z+1}^n}{p^{z+1}} \times$ $\overline{z+c}$; or by drawing it into p^{z+1}, and dividing by A, $p z^n \times \overline{z+a} = m \times$ $\overline{z+1}^n \times \overline{z+c}$. But by *Newton*'s theorem for evolving a binomial, $\overline{z+1}^n$ is $= z^n + n z^{n-1}$, and that accurately because z is infinitely great. Substitute this value, and the equation will be $p z^n \times \overline{z+a} = m \times \overline{z^n + n z^{n-1}} \times$ $\overline{z+c}$, which divided by z^{n-1}, becomes $p z \times \overline{z+a} = m \times \overline{z+n} \times \overline{z+c}$, or $p z z + p a z = m z z + \overline{n+c} \times m z + m n c$; in which, by comparing the homologous terms, it will be $p = m$, and $p a = \overline{n+c} \times m$, whence $n = a - c$; and therefore the last sum S, which before was $\frac{A z^n}{p^z}$, is now $\frac{A z^{a-c}}{m^z}$; where it is noted, the coefficient A is not determined. Suppose now $m = 1$, and $a = c$, the sum will become $\frac{A z^0}{z}$; but z^0 is $= 1$, and $1^z_1 = 1$, tho' z be infinitely great; and consequently the last of the sums is finite, for it is equal to A when $m = 1$, and also $a = c$: But there is no other case in which $\frac{A z^{a-c}}{m^z}$ is a finite quantity, when z is infinitely great; therefore the proposition is evident.

C O R O L L A R Y.

IF m be less than unity, the last sum will be infinitely great; but if greater, then m will be infinitely small; and if m be unity, that sum will be infinitely great or small according as a is greater or less than c. Therefore in the equation to the sums, if m be greater than unity, or equal to it, and at the same time a less than c, the last sum will always be nothing, and there will want no correction.

E X A M P L E.

SUPPOSE the first sum $A = 1$, the second $B = \frac{9}{7}A$, the third $C = \frac{2 \cdot 5}{1 \cdot 4}B$, the fourth $D = \frac{4 \cdot 9}{2 \cdot 7}C$, the fifth $E = \frac{8 \cdot 1}{1 \cdot 6}D$, &c. And the equation to the same will be $S \times \overline{z z + z + \frac{1}{4}} = S^I \times \overline{z z + z}$; which compar'd with the general equation, gives $m = 1$, $a = 1$, $c = 1$, and thence $a - c = 0$; whence by this proposition, the last of the sums, that is, the product under all the numbers $1 \times \frac{9}{7} \times \frac{2 \cdot 5}{1 \cdot 4} \times \frac{4 \cdot 9}{2 \cdot 7} \times \frac{8 \cdot 1}{1 \cdot 6} \times$ &c. *in infinitum*, is a finite quantity.

In the equation to the sums, substitute S—T for S^I, and you will find $S = -4 T \times \overline{z z + z}$, which, because it comes out negative in respect of T,

exhibits the fum of the terms not from a given term forwards *ad infinitum*, but from a given term backwards, or towards the beginning of the feries.

But that we may make the thing plainer, fuppofe there be propofed two equations, $Szz = S^I \times \overline{zz-1}$, and $Sz = S^I \overline{z+1}$; each of thefe will give the fame equation to the terms, namely, $Tz = T^I \overline{z+2}$; from the former is deduced $S = -T \times \overline{zz-1}$, and from the latter $S = T \overline{z+1}$. In the firft cafe S is the fum of the terms from the beginning to T, and in the fecond S is the fum of T and of all the following terms *in infinitum*. In the equation to the terms, let the numbers 1, 2, 3, 4, &c. be wrote fuccefsively for z, and let $\frac{1}{2}$ be the firft term, and there will come out this feries

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \text{ &c.}$$

where, if you wanted the fum of the four firft terms, write 5 for z in $T \times \overline{zz-1}$, the former value of the fum S, and the fifth term $\frac{1}{5.6}$ for T, and you will obtain $\frac{25-1}{5.6} = \frac{4}{5} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$, for the four firft terms; and if you would have the fum of all the terms, except thofe four, write 5 for z, and $\frac{1}{5.6}$ for T, in $T \overline{z+1}$, the latter value of S; and you will have $\frac{6}{5.6} = \frac{1}{5} = \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \frac{1}{8.9} + \text{ &c.}$ And thefe two values of S added together, that is, the fum of $\frac{4}{5}$ and $\frac{1}{5}$, makes $\frac{4+1}{5} = 1$ for the value of all the terms from the beginning *in infinitum*.

P R O P O S I T I O N   VII.

IF an equation to the feries be $\overline{z-n} T + \overline{m-1} . z T^I$, S will be $= \frac{m-1}{m}$ $T + \frac{n}{z} \times \frac{A}{m} + \frac{n+1}{z+1} \times \frac{B}{m} + \frac{n+2}{z+2} \times \frac{C}{m} + \frac{n+3}{z+3} \times \frac{D}{m} + \frac{n+4}{z+4} \times \frac{E}{m} + \text{ &c.}$

Imagine the fum S to be equal to the term T drawn into y, that is, $S = Ty$, then write $S-T$, T^I, y^I, the latter values of the indeterminate quantities, for their former values S, T, and y refpectively, and we fhall have $S-T = T^I y^I$, which taken from the former $S = Ty$, leaves $T = Ty - T^I y^I$, whence $T^I = T \times \frac{y-1}{y^I}$. But by the equation to the feries, namely, $\overline{z-n} T$

$+ m - 1$

$+ \overline{m-1}.z \mathrm{T}^{\mathrm{I}}$, it is $\mathrm{T}^{\mathrm{I}} = - \dfrac{\overline{z-n}\mathrm{T}}{z.m-1}$; therefore by making these two values

of T^{I} equal between themselves, it will be $\mathrm{T} \times \dfrac{y-1}{y^{\mathrm{I}}} = - \dfrac{\overline{z-n}\mathrm{T}}{z.m-1}$; which

divided by T, and drawn into $\overline{m-1}.y^{\mathrm{I}}$, is $\overline{m-1}.y - m + 1 = -y^{\mathrm{I}} + \dfrac{n}{z}y^{\mathrm{I}}$, or

$\overline{m-1}.y + y^{\mathrm{I}} - \dfrac{n}{z}y^{\mathrm{I}} - m + 1 = 0$; which is a differential equation, by the
resolution of which, y the root will be given. To that end assume $y = a +$
$\dfrac{b}{z} + \dfrac{c}{z.z+1} + \dfrac{d}{z.z+1.z+2} + \dfrac{e}{z.z+1.z+2.z+3}$ + &c. then for y and z
write their succeeding values y^{I} and $z+1$ respectively, and you will have
$y^{\mathrm{I}} = a + \dfrac{b}{z+1} + \dfrac{c}{z+1.z+2} + \dfrac{d}{z+1.z+2.z+3} + \dfrac{e}{z+1.z+2.z+3.z+4}$
+ &c. that is, $y^{\mathrm{I}} = a + \dfrac{b}{z} + \dfrac{c-b}{z.z+1} + \dfrac{d-2c}{z.z+1.z+2} + \dfrac{e-3d}{z+1.z+2.z+3}$
+ &c. And the first value of y^{I}, multiplied by $\dfrac{n}{z}$, gives $\dfrac{n}{z}y^{\mathrm{I}} = \dfrac{na}{z} +$
$\dfrac{nb}{z.z+1} + \dfrac{nc}{z.z+1.z+2} + \dfrac{nd}{z.z+1.z+2.z+3}$ + &c. Substitute these va-
lues, now reduced to the same form, in the equation, and there will result
$ma - m + 1 + \dfrac{mb-na}{z} + \dfrac{mc-\overline{n+1}.b}{z.z+1} + \dfrac{md-\overline{n+2}.c}{z.z+1.z+2}$ + &c. = 0, where by

putting the homologous terms equal to nothing, we shall have $a = \dfrac{m-1}{m}$,

$b = \dfrac{n}{m}a$, $c = \dfrac{n+1}{m}b$, $d = \dfrac{n+2}{m}c$, $e = \dfrac{n+3}{m}d$, &c. And these being given,
the value of y the root will be given; which, lastly, drawn into T will
give for S the series laid down in this proposition. $\mathcal{Q}. E. D.$

C O R O L L A R Y.

If n be a whole and negative number, or nothing, the value of S will
terminate, and the series will be summable. And when m is negative,
the series will be infinitely great, except when $m = 0$, for then the series
will be summed by the first example of the fifth proposition.

E X A M P L E   I.

SUPPOSE this series $\frac{1}{3} + \frac{2}{5}A + \frac{3}{7}B + \frac{4}{9}C + \frac{5}{11}D + \frac{6}{13}E +$ &c. were propos-
ed

ed to be summed, the equation defining it, is $\overline{z-\frac{1}{4}}T-2zT^I=0$, where the successive values of z are $1\frac{1}{4}$, $2\frac{1}{4}$, $3\frac{1}{4}$, $4\frac{1}{4}$, &c. and the series being compared with the general equation, gives $n=\frac{1}{2}$, $m-1=-2$, or $m=-1$; which being substituted, there arises

$$S=2T-\frac{A}{2z}-\frac{3B}{2z+2}-\frac{5C}{2z+4}-\frac{7D}{2z+6}-\frac{9E}{2z+8}- \&c.$$

In which if there be substituted any term for T, and for z its corresponding value, S will be the sum of T and of all the following terms *in infinitum*. Wherefore, I gather 12 of the first terms, and their sum comes out .78533961813; then to obtain the sum of the rest, I write the thirteenth term, that is, 00003029411 for T, and for z its proper value $13\frac{1}{4}$; and I have $S=.0000605\,8822-\frac{1}{27}A-\frac{3}{35}B-\frac{5}{11}C-\frac{7}{77}D-\frac{9}{35}E-$ &c. When the terms come out negative and affirmative alternately, I dispose them separately in two columns, as under;

+.000060 58822	—.0000022440 1
+ - - - - 23214	— - - - - - 3744
+ - - - - - - 794	— - - - - - - 204
+ - - - - - - 61	— - - - - - - 20
+ - - - - - - - 7	— - - - - - - 3
+ - - - - - - - 1	— - - - - - - - 1
+.000060 82899	—.00000228373

And the sum of the negatives .00000228373, taken from the sum of the affirmatives .00006082899, leaves .00005854526=S, which added to the sum of the first terms exhibits .78539816339 for the value of the proposed series, that is, for the area of a circle whose diameter is unity.

E X A M P L E II.

REQUIRED the value of this series $1-\frac{1}{2}A-\frac{1}{4}B-\frac{1}{6}C-\frac{7}{8}D-\frac{9}{10}E-$ &c. The equation defining the relation of the terms is $\overline{z-\frac{1}{2}}T+zT^I=0$; wherein the values of z the abscissa are $1.2.3.4$, &c. and by comparing this equation with that in the theorem, n will be $=\frac{1}{2}$, $m-1=1$, and $m=2$; and consequently

$$S=\frac{1}{2}T+\frac{A}{4z}+\frac{3B}{4z+4}+\frac{5C}{4z+8}+\frac{7D}{4z+12}+\frac{9E}{4z+16}+ \&c.$$

Now add together ten of the first terms of the series to be transformed, and you will find their sum to be .6168670654; lastly, to get the sum

L   of

of the reſt, in the value of S write the thirteenth term, *viz.* .1761970520 for T, and 11 for z; and there will come out S=
.0880985260+$\frac{1}{11}$A+$\frac{3}{12}$B+$\frac{5}{13}$C+$\frac{7}{14}$D+$\frac{9}{15}$E+ &c. and by making the computation as in the margin, you will find S=.0902397156, which added to the ag-aggregate of the firſt terms before found, makes .7071067810 for the value of the ſeries, that is, for $\sqrt{\frac{1}{2}}$. For $\sqrt{\frac{1}{2}}$ may be wrote thus $\overline{1+1}^{-\frac{1}{2}}$, and e-volved according to *Newton*'s theorem, the ſeries is that we have now mentioned.

.0880985260
20022392
1251400
120327
15041
2256
388
74
15
3
S= .0902397156

S C H O L I U M.

Every ſeries whoſe terms are alternately affirmative and negative, if it be transformed by this propoſition, will paſs into another converging ſwifter, whoſe terms are of the ſame ſign. And, *vice verſâ*, every ſeries, whoſe terms are of the ſame ſign, will go into another, whoſe terms are alternately negative and affirmative; which will not yet converge ſwifter than the former, except when the transformation is begun from terms ſufficiently remote from the beginning. And if the ſeries be transformed, and that which comes out after the firſt transformation, be again tranſ-form'd, it will be the ſame as that firſt propoſed. For *example*, ſuppoſe this ſeries 1−$\frac{1}{3}$+$\frac{1}{5}$−$\frac{1}{7}$+$\frac{1}{9}$− &c. is to be transformed, there will come out $\frac{1}{2}$+$\frac{1}{3}$A+$\frac{1}{5}$B+$\frac{1}{7}$C+$\frac{1}{9}$D+ &c. And again, if this laſt be transform-ed, the firſt will come out 1−$\frac{1}{3}$+$\frac{1}{5}$−$\frac{1}{7}$+ &c. that is, it transfers the ſeries from the tangent to the ſine, and from the ſine to the tangent; but in theſe caſes the operation muſt begin at the firſt term of the ſeries, namely, when you are about the transformation of the whole.

P R O P O S I T I O N   VIII.

IF an equation to the ſeries be $T^1=\frac{z-m}{z}\times\frac{z-n}{z-n+1}T$, *and* A *be put* = $\overline{z-n}T$, B=$\frac{n}{z}$A, C=$\frac{n+1}{z+1}$B, D=$\frac{n+2}{z+2}$C, E=$\frac{n+3}{z+3}$D, &c. S *will be* =$\frac{A}{m}+\frac{B}{m+1}+\frac{C}{m+2}+\frac{D}{m+3}$, &c.

For ſuppoſe S=T×$\overline{z-n}$×y, and by the differential method S−T will be =$T^1\times\overline{z-n+1}\times y^1$, the difference of theſe equations will give T=T× $\overline{z-n}\times y-T^1\times\overline{z-n+1}\times y^1$. For T^1 write its value, namely $\frac{z-m}{z}\times$
$z-n$

$\frac{z-n}{z-n+1}$T, we shall have $T = T \times \overline{z-n} \times y - T \times \frac{z-m}{z} \times \overline{z-n} \times y^I$, which

divided by $T \times \overline{z-n}$, becomes $\frac{1}{z-n} = y - \frac{z-m}{z} y^I$, or $y - y^I + \frac{m}{z} y^I - \frac{1}{z-n} = 0$.

From this equation the root y will be found after the following manner.

Suppose $y = \frac{a}{m} + \frac{b}{m+1 \cdot z} + \frac{c}{m+2 \cdot z \cdot z+1} + \frac{d}{m+3 \cdot z \cdot z+1 \cdot z+2}$
$+$ &c. then will

$y^I = \frac{a}{m} + \frac{b}{m+1 \cdot z+1} + \frac{c}{m+2 \cdot z+1 \cdot z+2} + \frac{d}{m+3 \cdot z+1 \cdot z+2 \cdot z+3}$
$+$ &c. And

$y - y^I = \frac{b}{m+1 \cdot z \cdot z+1} + \frac{2c}{m+2 \cdot z \cdot z+1 \cdot z+2} + \frac{3d}{m+3 \cdot z \cdot z+1 \cdot z+2 \cdot z+3}$
$+$ &c. And

$\frac{m}{z} y^I = \frac{am}{mz} + \frac{bm}{m+1 \cdot z \cdot z+1} + \frac{cm}{m+2 \cdot z \cdot z+1 \cdot z+2} + \frac{dm}{m+3 \cdot z \cdot z+1 \cdot z+2 \cdot z+3}$
$+$ &c.

Consequently $y - y^I + \frac{m}{z} y^I = \frac{a}{z} + \frac{b}{z \cdot z+1} + \frac{c}{z \cdot z+1 \cdot z+2} + \frac{d}{z \cdot z+1 \cdot z+2 \cdot z+3}$
$+$ &c.

But $\frac{1}{z-n}$ is $= \frac{1}{z} + \frac{n}{z \cdot z+1} + \frac{n \cdot n+1}{z \cdot z+1 \cdot z+2} + \frac{n \cdot n+1 \cdot n+2}{z \cdot z+1 \cdot z+2 \cdot z+3} +$ &c.

Therefore $y - y^I + \frac{m}{z} y^I - \frac{1}{z-n} = \frac{a-1}{z} + \frac{b-n}{z \cdot z+1} + \frac{c-n \cdot n+1}{z \cdot z+1 \cdot z+2} +$

$\frac{d-n \cdot n+1 \cdot n+2}{z \cdot z+1 \cdot z+2 \cdot z+3} +$ &c. $= 0$.

Now put the numerators $= 0$, that the terms may vanish; and for determining the coefficients you will have the following equations $a = 1$, $b = n$, $c = n \cdot \overline{n+1}$, $d = n \cdot \overline{n+1} \cdot \overline{n+2}$, &c. Substitute these values in the series assumed for y, instead of a, b, c, d, &c. and the value of y, arising from thence drawn into $\overline{z-n}$T, will exhibit for S, the series placed in the theorem. *Q. E. D.*

C O R O L L A R Y.

If n be an integer and negative, or nothing, the series will be summed accurately by this theorem; and if m be nothing or negative, the series will be infinitely great. This proposition and the forgoing one are of use for the quadrature of binomial curves. This is of use when in the ordinate $x^\theta \times \overline{e+fx^n}|^\rho$, $e+fx^n$ is $= 0$. And the former is to be used when the contrary happens.

E X-

E X A M P L E   I.

REQUIRED the value of this series $1 + \frac{1.1}{2.3}A + \frac{3.3}{4.5}B + \frac{5.5}{6.7}C + \frac{7.7}{8.9}D + \frac{9.9}{10.11}E +$ &c.

The equation defining it is $T^1 = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{1}{2}}{z + \frac{1}{2}}T$, as will appear by writing the values 1, 2, 3, 4, &c. fucceffively for z. But the equation in the theorem compared with this, gives $m = \frac{1}{2}$, $n = \frac{1}{2}$, whence $A = \overline{z - \frac{1}{2}}T$, $B = \frac{A}{2z}$, $C = \frac{3B}{2z+2}$, $D = \frac{5C}{2z+4}$, $E = \frac{7D}{2z+6}$, &c. And $S = \frac{1}{1}A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D + \frac{1}{9}E +$ &c. By performing the computation I find the fum of twelve terms to be 1.407397508. Then I fubftitute the thirteenth term, that is, $\frac{1}{13} \times .161180258$ for T, and for z, 13 its correfponding value; and there comes out

$A = \frac{1}{1} \times .161180258$	.161180258
$B = \frac{1}{3} \times \quad\ \ 6199241$	2066414
$C = \frac{1}{5} \times \ \ -\ \ 664204$	132841
$D = \frac{1}{7} \times \ ---\ 110701$	15814
$E = \frac{1}{9} \times \ ---\ 24261$	2691
$F = \frac{1}{11} \times \ ----\ 6410$	583
$G = \frac{1}{13} \times \ ----\ 1959$	151
$H = \frac{1}{15} \times \ ----\ 670$	45
$I = \frac{1}{17} \times \ -----\ 250$	15
$K = \frac{1}{19} \times \ -----\ 102$	5
$L = \frac{1}{21} \times \ -----\ 44$	2
$M = \frac{1}{23} \times \ -----\ 20$	1
	$S = .163398820$

From this computation I get $S = .163398820$, for the value of all the terms after the twelfth, which therefore I add to the fum of the firft terms, 1.407397508, and I get 1.570796328 for the value of the feries to be fummed, that is, for the length of a femicircular arc whofe diameter is unity.

The initial terms are eafily reduced into decimal fractions, by the following rule, when their fum is found, viz. put $a = 1$, $b = a - \frac{1}{4}a$, $c = b - \frac{3}{4}b$, $d = c - \frac{5}{6}c$, $e = d - \frac{7}{8}d$, &c. and the terms will be a, $\frac{1}{3}b$, $\frac{1}{5}c$, $\frac{1}{7}d$, $\frac{1}{9}e$, &c.

EXAMPLE

EXAMPLE II.

LET this series of my lord *Brounker* be proposed, $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} +$ &c.

The equation answering to the same is $T^I = \frac{z-1}{z} \times \frac{z - \frac{1}{2}}{z + \frac{1}{2}} T$, or likewise $T^I = \frac{z-1}{z} \times \frac{z - \frac{1}{2}}{z - \frac{1}{2}} T$, namely by taking the beginning of the abscissa z from diverse points; in the first the values of z are $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, &c. $m = 1$, $n = \frac{1}{2}$, $A = \overline{z - \frac{1}{2}} T$, $B = \frac{A}{2z}$, $C = \frac{3B}{2z+2}$, $D = \frac{5C}{2z+4}$, &c. In the latter the values of z are, 2, 3, 4, 5, &c. $m = 1$, $n = \frac{1}{2}$, $A = \overline{z - \frac{1}{2}} T$, $B = \frac{3A}{2z}$, $C = \frac{5B}{2z+2}$, $D = \frac{7C}{2z+4}$, &c. and in both, S will be $= A + \frac{1}{2} B + \frac{1}{3} C + \frac{1}{4} D +$ &c. Therefore S will be given to more figures, by assuming for T any of the terms sufficiently distant from the beginning. And likewise in other cases we may institute a *computus*, and that by two different ways, when the terms are assignable. But this series, wrote after this manner, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c. will be easier managed by the third or seventh proposition.

SCHOLIUM.

HITHERTO we have treated of the summing up of series which are produced in the quadrature of binomial curves, and the like. But we might proceed in the same manner in cases that are more difficult; for the sum of the series is determined, and may be found from the given relation of the terms; and that by resolving a differential equation, as in the last two propositions. But the problem would be laborious to seek the sums independently of the terms, when the terms are not assignable; consequently I have sought a quantity, which drawn into the term T exhibits the sum S. So likewise the areas of curves are easily found by the middle ordinates, for series coming out this way are the most simple. And these being premis'd, I proceed to my method of resolving the roots of differential equations, in series indeed more compounded, but at the same time far more converging than the former; which, for this cause only, I have exhibited, that they were simple, and fit for common purposes.

Hitherto we have denoted any sum by S, and its terms by T, T^I, T^{II}, &c. but in the following we shall denote the series or sums, by S_2, S_3, S_4,

&c. and

&c. and their terms by T_2, T^I_2, T^{II}_2, &c. T_3, T^I_3, T^{II}_3, &c. Consequently

$$S_2 = T_2 + T^I_2 + T^{II}_2 + \&c.$$
$$S_3 = T_3 + T^I_3 + T^{II}_3 + \&c.$$
$$S_4 = T_4 + T^I_4 + T^{II}_4 + \&c.$$

And as in the series S, the succesfive fums are denoted by S^I, S^{II}, &c. fo in the series S_2, they will be denoted by S^I_2, S^{II}_2, &c. and in the series S_3, by S^I_3, S^{II}_3, &c. and fo in others. But this notation I was forced to introduce, becaufe the fums and terms of different feries come to be confidered together in the equation.

P R O P O S I T I O N   IX.

G*IVEN the relation between two fums in different feries, and the equation to the terms in either of the two; to find the equation to the terms in the other.*

The problem is folved by proceeding from the prefent relation of the variable quantities to that which fucceeds, that the fums may from thence be expunged, which will be manifeft from the following examples.

E X A M P L E   I.

Suppose S and S_2 be two fums in different feries, and their Relation $S = \overline{\dfrac{m+1.z-n}{m.m+1}}T + S_2$; but let $T^I = \dfrac{z-m}{z} \times \dfrac{z-n}{z-n+1}T$, be the relation of the terms of the feries S; and from thefe given required to find the equation to the terms of the feries S_2. In the equation $S = \overline{\dfrac{m+1.z-n}{m.m+1}}T + S_2$, exhibiting the relation of the fums, fubftitute the confequent values of the variable quantities inftead of the preceding ones, that is, S^I, or $S-T$, for S; S^I_2, or $S_2 - T_2$, for S_2; T^I for T, and $z+1$ for z; and you will have $S-T = \overline{\dfrac{m+1.z+1-n}{m.m+1}}T^I + S_2 - T_2$, which taken from the former leaves

the equation freed from the fums, namely, $T = \overline{\dfrac{m+1.z-n}{m.m+1}}T - \overline{\dfrac{m+1.z+1-n}{m.m+1}}T^I + T_2$; whence T_2 is $= \overline{\dfrac{m+1.z+1-n}{m.m+1}}T^I - \overline{\dfrac{m+1.z-m-n}{m.m+1}}T$, in which fubftitute for T^I its own value $\dfrac{z-m}{z} \times \dfrac{z-n}{z-n+1}T$, and you

<div align="right">will</div>

will have $T_2 = \dfrac{n \cdot \overline{m-n+1}}{m+1 \cdot z \cdot \overline{z-n+1}} T$. Moreover in this value write the succeeding values T'_2, T', and $z+1$, of the indeterminate quantities, for the present T_2, T, and z, and there will arise $T'_2 = \dfrac{n \cdot \overline{m-n+1}}{m+1 \cdot \overline{z+1} \cdot \overline{z-n+2}} T'$; where for T' substituting again its value, it becomes $T'_2 = \dfrac{n \cdot \overline{m-n+1}}{m+1 \cdot \overline{z+1} \cdot \overline{z-n+2}} T \times \dfrac{z-m}{z} \times \dfrac{z-n}{z-n+1}$, from which there comes out $T = \dfrac{m+1 \cdot z \cdot \overline{z+1} \cdot \overline{z-n+1} \cdot \overline{z-n+2}}{n \cdot m-n+1 \cdot \overline{z-m} \cdot \overline{z-n}} T'_2$. But the equation $T_2 = \dfrac{n \cdot \overline{m-n+1}}{m+1 \cdot z \cdot \overline{z-n+1}} T$, before found, gives $T = \dfrac{m+1 \cdot z \cdot \overline{z-n+1}}{n \cdot m-n+1} T_2$. Now let the two values of the term T be made equal to one another, and you will get this equation, $T'_2 = \dfrac{z-m}{z+1} \times \dfrac{z-n}{z-n+2} T_2$, which expresses the relation of the terms of the series S_2. _Q. E. I._

E X A M P L E II.

Suppose now the relation between the sums $S = \frac{1}{4} \times \dfrac{4z+3}{4z+2} T + S_2$, and $zT + 3T' \times \overline{z+1}$ the equation to the series S. In the equation to the sums write the successive values of the variable quantities, you will have $S - T = \frac{1}{4} \times \dfrac{4z+7}{4z+6} T' + S_2 - T_2$; which taken from the former, you will have $T = \frac{1}{4} \times \dfrac{4z+3}{4z+2} T - \frac{1}{4} \times \dfrac{4z+7}{4z+6} T' + T_2$; from which you will find $T_2 = \frac{1}{4} \times \dfrac{4z-1}{4z+2} T + \frac{1}{4} \times \dfrac{4z+7}{4z+6} T'$. But by the equation to the series S, $3T' = -\dfrac{z}{z+1} T$; which being wrote, there comes out $T_2 = +\frac{1}{4} \times \dfrac{4z-1}{4z+2} T + \frac{1}{3} \times \dfrac{4z+7}{4z+6} \times \dfrac{-z}{z+1} T$, or $T_2 = -\frac{1}{4} T \times \dfrac{1}{2z+1} \times \dfrac{1}{2z+2} \times \dfrac{1}{2z+3}$. And again, by having recourse to the succeeding values of the indeterminate quantities, we shall have $T'_2 = -\frac{1}{4} T' \times \dfrac{1}{2z+3} \times \dfrac{1}{2z+4} \times \dfrac{1}{2z+5}$, or by substituting $-\dfrac{z}{z+1} T$ for $3T'$, $T'_2 = \frac{z}{3} \times \dfrac{1}{2z+2} \times \dfrac{1}{2z+3} \times \dfrac{1}{2z+4} \times \dfrac{1}{2z+5} T$; whence $T = \frac{3}{z} \cdot \overline{2z+2} \cdot \overline{2z+3} \cdot \overline{2z+4} \cdot \overline{2z+5} \cdot T'_2$; and by the value of T_2, the same

same T is $= -\frac{1}{3} \times \overline{2z+1} \times \overline{2z+2} \times \overline{2z+3}\,T2$; which two values being made equal betwixt themselves, we shall obtain $\overline{zz+\frac{1}{4}z} \times T2 + \overline{zz+\frac{9}{4}z+5} \times 3T^1 2 = 0$, for the equation to the terms of the series S2. And in like manner we must proceed in other cases.

S C H O L I U M.

By this proposition, infinite series may be compared amongst themselves. For an equation exhibiting the relation between the sums S and S2, will give one from the other being given; and the terms of S and S2, will be given from their own equations, whereof one is assumed, but the other is found from the assumed; as in the above examples of this proposition. As if the relation of the sums be $S = \frac{2z-1}{2}T + S2$, and the equation to the series S be $T^1 = \frac{zz-2z+1}{zz}T$, you will find $T2 = \frac{T}{2zz}$, and $T^1 2 = \frac{zz-2z+1}{zz+2z+1}T2$, the equation to the series S2. Let now 2, 3, 4, 5, &c. be the succeeding values of the indeterminate quantity z, and by putting unity for T, you will find

$$S = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \tfrac{1}{36} + \&c.$$
$$S2 = \tfrac{1}{8} \text{ into } 1 + \tfrac{1}{9} + \tfrac{1}{36} + \tfrac{1}{100} + \tfrac{1}{225} + \&c.$$

In the first, the denominators are squares of the natural numbers, and in the last, the squares of the triangular numbers. But the relation between the series $S = \frac{2z-1}{2}T + S2$ (by writing 2 for z, and unity for T) will give $S2 = S - \frac{3}{4}$; and by drawing it into 8, will be $8S - 12 = 8S2 = 1 + \frac{1}{9} + \frac{1}{36} + \frac{1}{100} + \frac{1}{225} + \&c.$ And likewise the series whose denominators are the square of pyramidal numbers, triangulo-triangulars, &c. may be compared with a series in which the denominators are squares of the natural numbers; and, in general, all series may be compared which are defined by the following equations,

$$T^1 = \frac{zz+a}{\overline{z+b}.\overline{z+c}}T, \quad T^1 = \frac{zz+a}{\overline{z+b+1}.\overline{z+c+1}}T, \quad T^1 = \frac{zz+a}{\overline{z+b+2}.\overline{z+c+2}}T,$$

&c. where the numerator remains the same, but the denominators are formed by writing continually $z+1$ for z.

P R O P O S I T I O N X.

To find, nearly, the value of a series defined by an equation of this form,
$$T \times \overline{z^\theta + az^{\theta-1} + bz^{\theta-2} + \&c.} = rT^1 \times \overline{z^\theta + cz^{\theta-1} + dz^{\theta-2} + \&c.}$$

C A S E I.

First let $r = 1$, and let us suppose the equation to the series be $T^1 = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$. Let us put $S = \frac{z+p}{q} T$ *quam proximè*, or $S = \frac{z+p}{q} T + S_2$ accurately; and by proceeding to the succeeding values of the indeterminate quantities, $S - T$ will be $= \frac{z+p+1}{q} T^1 + S_2 - T_2$; the difference of these equations will give $T = \frac{z+p}{q} T - \frac{z+p+1}{q} T^1 + T_2$, whence $T_2 = \frac{z+p+1}{q} T^1 - \frac{z+p-q}{q} T$. For T^1 substitute its value $\frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, and there will come out $T_2 = \frac{z+p+1}{q} \times \frac{z-m}{z} \times \frac{z-n}{z-n+1} T - \frac{z+p-q}{q} T$; which reduced to a common denominator, gives

$$T_2 = \frac{\overline{q-m} \times zz + \overline{m-q} \times \overline{n-1} \atop -n-p \times \overline{m+1} \Big\} z + m \cdot n \cdot \overline{p+1}}{q \cdot z \cdot z - n + 1} T.$$

Now, the lesser the sum S_2 is, the nearer will $\frac{z+p}{q} T$ accede to the value of S; and the lesser S_2 shall be, the lesser is its first term T_2; but it will be the least, when the variable quantity z is of the least dimensions in the numerator of its value; for z here is supposed to be great; therefore put the coefficients of the powers z^2 and z, equal to nothing, and you will have two equations $q - m = 0$, and $\overline{m-q} \times \overline{n-1} - n - p \times \overline{m+1} = 0$, to determine the two assumed quantities p, q. The first equation gives $q = m$, from which and the second, is found $p = -\frac{n}{m+1}$; whence T_2 will be $= \frac{n \cdot \overline{m-n+1}}{m+1 \cdot z \cdot z - n + 1} T$, and $S = \frac{\overline{m+1} \cdot z - n}{m \cdot \overline{m+1}} T$ *ferè*. 2. E. I.

And in like manner will be found the approximation, when the equation to the series is $T^1 = \frac{zz + az + b}{zz + cz + d} T$, or more compounded; but the coefficients after b and d do not come in this computation.

C A S E II.

Suppose now the equation be $T \times \overline{zz + az + b} + rT^1 \times \overline{zz + cz + d} = 0$, r being

r being any number, except unity. Suppose $S = p \times \frac{z+m}{z+n} T + S_2$; then write the latter values of the variable quantities for the former, and it will become $S - T = p \times \frac{z+m+1}{z+n+1} T^1 + S_2 - T_2$, which taken from the former, leaves $T = p \times \frac{z+m}{z+n} T - p \times \frac{z+m+1}{z+n+1} T^1 + T_2$; this equation will give $T_2 = \frac{z - pz - mp + n}{z+n} T + p \times \frac{z+m+1}{z+n+1} T^1$; but by the equation to the series S, T^1 is $= -\frac{T}{r} \times \frac{zz + az + b}{zz + cz + d}$, which being wrote for T^1, there comes out $T_2 = \frac{z - pz - mp + n}{z+n} T - \frac{pT}{r} \times \frac{z+m+1}{z+n+1} \times \frac{zz + az + b}{zz + cz + d}$. If the terms of this value be reduced to a common denominator, and the coefficients of the three highest powers of z be put equal to nothing, the first equation will give $rp + p = r$, the second $m - n \times r + 1 = c - a$, the third $c - a \times 2n + 1 + d - b = m - n \times rc + rn + r + n + ma$; and these three equations give $p = \frac{r}{r+1}$, $m = c - \frac{b-d}{a-c} - \frac{1}{r+1}$, and $n = m + \frac{a-c}{r+1}$; wherefore the assumed quantities p, m, and n, are given; and from thence the quantity $p \times \frac{z-m}{z+n} T$, which is equal to the series S, very nearly. *Q.E.I.*

P R O P O S I T I O N   XI.

IF $T^1 = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$ *be an equation exhibiting the relation of the terms of the series* S, *put*

$$T_2 = \frac{1}{m} \times \frac{m}{m+1} \times \frac{n}{z} \times \frac{m-n+1}{z-n+1} T,$$

$$T_3 = \frac{2}{m+2} \times \frac{m+1}{m+3} \times \frac{n+1}{z+1} \times \frac{m-n+2}{z-n+2} T_2,$$

$$T_4 = \frac{3}{m+4} \times \frac{m+2}{m+5} \times \frac{n+2}{z+2} \times \frac{m-n+3}{z-n+3} T_3,$$

$$T_5 = \frac{4}{m+6} \times \frac{m+3}{m+7} \times \frac{n+3}{z+3} \times \frac{m-n+4}{z-n+4} T_4,$$

$$T_6 = \frac{5}{m+8} \times \frac{m+4}{m+9} \times \frac{n+4}{z+4} \times \frac{m-n+5}{z-n+5} T_5,$$

&c.

And

And S will be $= \dfrac{\overline{m+1}.\overline{z-1}.n}{m.\overline{m+1}}T +$

$$\dfrac{\overline{m+3}.\overline{z+2}-2.\overline{n+1}}{\overline{m+2}.\overline{m+3}}T2 +$$

$$\dfrac{\overline{m+5}.\overline{z+4}-3.\overline{n+2}}{\overline{m+4}.\overline{m+5}}T3 +$$

$$\dfrac{\overline{m+7}.\overline{z+6}-4.\overline{n+3}}{\overline{m+6}.\overline{m+7}}T4 +$$

$$\dfrac{\overline{m+9}.\overline{z+8}-5.\overline{n+4}}{\overline{m+8}.\overline{m+9}}T5 +$$

&c.

By the preceding propofition, the quantity $\dfrac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}T$ is equal to the feries S nearly. Therefore let $S = \dfrac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}T + S2$ accurately, and you will find $T2 = \dfrac{1}{m} \times \dfrac{m}{m+1} \times \dfrac{n}{z} \times \dfrac{m-n+1}{z-n+1}T$, and $T'2 = \dfrac{z-m}{z+1} \times \dfrac{z-n}{z-n+1}T2$ the equation to the terms of the feries S2; from which being given, the quantity $\dfrac{\overline{m+3}.\overline{z+2}-2.\overline{n+1}}{\overline{m+2}.\overline{m+3}}T2$, by the above propofition, is found nearly equal to the feries S2. Then by affuming $S2 = \dfrac{\overline{m+3}.\overline{z+2}-2.\overline{n+1}}{\overline{m+2}.\overline{m+3}}$ $T2 + S3$, you will find $T3 = \dfrac{2}{m+2} \times \dfrac{m+1}{m+3} \times \dfrac{n+1}{z+1} \times \dfrac{m-n+2}{z-n+2}T2$; and $T'3 = \dfrac{z-m}{z+2} \times \dfrac{z-n}{z-n+3}T3$, an equation to the terms of the feries S3; whence you will find the quantity $\dfrac{\overline{m+5}.\overline{z+4}-3.\overline{n+2}}{\overline{m+4}.\overline{m+5}}T3$ nearly equal to the feries S3; and fo we might proceed further. Therefore the firft term in the value of S is equal to S nearly, and the fecond term equal to S2 nearly, and the third equal to S3 nearly, and fo in the reft, that is, the firft term is nearly equal to the feries whofe value is fought; the fecond is nearly equal to the defect of the firft term from the true one; the third is nearly equal to the defect of the two firft from the true one; the fourth is equal nearly to the defect of the three firft from the true one, &c. Therefore the value of the fum S is a true one, and converges very faft. *Q.E.D.*

C O R -

C O R O L L A R Y.

THE value of the series S here exhibited will terminate when n, or $m-n+1$, is nothing, or a whole and negative number; and in other cases it will proceed *in infinitum* converging very fast to the truth, except when (because m is nothing, or a negative quantity) the value of the series is infinitely great.

E X A M P L E   I.

REQUIRED the value of this series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$ &c. The equation defining the relation of the terms is $T^1=\frac{z-1}{z}\times\frac{z-1}{z}T$; 2, 3, 4, 5, &c. being the values of the indeterminate quantity z. And this compared with the equation in the theorem, gives $m=1$, $n=1$, which being wrote out there comes out

$$T_2=\frac{T}{2z^2},\ T_3=\frac{8T_2}{6.\overline{z+1}|^2},\ T_4=\frac{27T_3}{10.\overline{z+2}|^2},\ T_5=\frac{64T_4}{14.\overline{z+3}|},\ \&c.$$

And $S=\frac{2z-1}{2}T+\frac{2z+2}{6}T_2+\frac{2z+5}{10}T_3+\frac{2z+8}{14}T_4+\frac{2z+11}{18}T_5+$&c.

Now collect the first ten terms, and you will find their aggregate or sum 1.5497.6773.1166.5406.9; then for z write its eleventh value, or 12; and the eleventh term $\frac{1}{144}$ for T; and you will find, by the computation,

T =.0082.6446.2809.9173.55	.0950.4132.2314.0495.8
T2= - - - 2869.6051.4233.24	1.2434.9556.1677.4
T3= - - - - 22.6398.8277.97	65.6556.6006.1
T4= - - - - - - 3118.7593.62	7128.5928.3
T5= - - - - - - - - 63.3652.70	123.2102.5
T6= - - - - - - - - - 1.7188.93	2.9690.0
T7= - - - - - - - - - - 583.96	920.9
T8= - - - - - - - - - - 23.78	34.9
T9= - - - - - - - - - - - 1.12	1.5
	S=.0951.6633.5681.6857.4

From this computation we have S=.0951.6633.5681.6857.4, which added to the sum of the first terms, exhibits 1.6449.3406.6848.2264.3 for the value of the series proposed.

E X A M P L E   II.

LET this series $1+\frac{1.1}{2.3}A+\frac{3.3}{4.5}B+\frac{5.5}{6.7}C+\frac{7.7}{8.9}D+\frac{9.9}{10.11}E+$ &c. be given to be summed.   The

The equation denoting the same is $T^I = \dfrac{z - \frac{1}{2}}{z} \times \dfrac{z - \frac{1}{2}}{z + \frac{1}{2}} T$, as will appear by writing the values 1, 2, 3, 4, &c. successively for z; and the equation in the theorem, comparing with this, gives $m = \frac{1}{2}$, $n = \frac{1}{2}$; consequently

$$T_2 = \frac{2 \cdot 1 \cdot 1}{1 \cdot 3 \cdot z \cdot 2z + 1} T, \quad T_3 = \frac{2 \cdot 4 \cdot 9}{5 \cdot 7 \cdot z + 1 \cdot 2z + 3} T_2, \quad T_4 = \frac{2 \cdot 9 \cdot 25}{9 \cdot 11 \cdot z + 2 \cdot 2z + 5} T_3,$$

And $S = \dfrac{6 \cdot z - 1 \cdot 2}{1 \cdot 3} T + \dfrac{14 \cdot \overline{z + 2} - 3 \cdot 4}{5 \cdot 7} T_2 + \dfrac{22 \cdot \overline{z + 4} - 5 \cdot 6}{9 \cdot 11} T_3 +$ &c.

By addition you will find the sum of ten of the first terms to be 1.3916.9464.5943.2880.5; then to find the sum of the rest, write 11 for z, and the eleventh term for T, and you will have

T $=$ .0083.9003.5809.6168.15	.1789.9383.0605.1587.3
T$_2=$ - - - 2210.8921.7644.71	1.0738.6191.4274.3
T$_3=$ - - - - - 15.1604.0349.56	45.9406.1665.3
T$_4=$ - - - - - - - 1963.2742.16	4570.9051.0
T$_5=$ - - - - - - - - - 38.8835.92	76.0818.3
T$_6=$ - - - - - - - - - 1.0484.94	1.8104.4
T$_7=$ - - - - - - - - - - - 358.18	562.5
T$_8=$ - - - - - - - - - - - - 14.77	21.5
T$_9=$ - - - - - - - - - - - - - 71	1.0
	S $=\overline{1791.0168.0851.6085.6}$

which added to S, the sum of the initial terms, there will come out for the value of the series 1.5707.9632.6794.8966.1, that is, for the semi-periphery of a circle whose diameter is unity.

E X A M P L E III.

Let this series $\frac{1}{1} + \frac{1 \cdot 3}{2 \cdot 7} A + \frac{3 \cdot 7}{4 \cdot 11} B + \frac{5 \cdot 11}{6 \cdot 15} C + \frac{7 \cdot 15}{8 \cdot 19} D$, &c. be now proposed; which is defined by this equation $T^I = \dfrac{z - \frac{1}{2}}{z} \times \dfrac{z - \frac{1}{4}}{z + \frac{1}{4}} T$, in which 1, 2, 3, 4, &c. are the values of the indeterminate quantity z. And m will be $= \frac{1}{2}$, $n = \frac{1}{4}$; and therefore $T_2 = \dfrac{5 \cdot 1 \cdot 1}{6 \cdot z \cdot 4z + 3} T$, $T_3 = \dfrac{9 \cdot 2 \cdot 3}{14 \cdot z + 1 \cdot 4z + 7} T_2$, $T_4 = \dfrac{13 \cdot 3 \cdot 5}{22 \cdot z + 2 \cdot 4z + 11} T_3$, and $S = \dfrac{6 \cdot z - 1 \cdot 1}{1 \cdot 3} T + \dfrac{14 \cdot \overline{z + 2} - 2 \cdot 5}{5 \cdot 7} T_2 + \dfrac{22 \cdot \overline{z + 4} - 3 \cdot 9}{9 \cdot 11} T_3 +$ &c.

O

Now add nine of the firſt terms together, and the ſum is .5055.0041.4718.3195.8 ; and writing 10 for z, and the tenth term for T, I have

T $=$.0047.5565.5924.4791.67	.0935.2789.9848.0902.9
T2$=$ - - - - 921.6387.4505.41	4160.5406.2053.0
T3$=$ - - - - - - 6.8760.0058.48	19.5167.2893.3
T4$=$ - - - - - - - 995.8557.00	2185.7755.9
T5$=$ - - - - - - - - 22.0991.75	40.9826.8
T6$=$ - - - - - - - - - 6653.41	1.0937.5
T7$=$ - - - - - - - - - - 252.54	379.0
T8$=$ - - - - - - - - - - 11.51	16.1
T9$=$ - - - - - - - - - - - 61	8
	S$=$.0935.6970.2649.4765.3

Laſtly, the ſum of the initial terms, added to S, makes .5990.7011.7367.7961.1 for the value of the ſeries, that is, for the ordinate of the elaſtic curve. And this number *James Bernouilli* hath rightly found to be contained between the limits .5983 and .6004.

E X A M P L E IV.

Let this Series $1 + \frac{1.1}{2.5}A + \frac{3.5}{4.9}B + \frac{5.9}{6.13}C + \frac{7.13}{10.21}E +$ &c. be propoſed; which is defined by this equation $T^I = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{1}{4}}{z + \frac{1}{4}} T$, 1, 2, 3, 4, &c. being the ſucceeding values of the indeterminate quantity z; here $m = \frac{1}{2}$, $n = \frac{1}{4}$, and conſequently

$$T2 = \frac{1.1.3}{2.z.4z + 1}T, \quad T3 = \frac{3.2.7}{10.z + 1.4z + 5}T2, \quad T4 = \frac{5.3.11}{18.z + 2.4z + 9}T3,$$

and $S = \frac{2z - 1}{1}T + \frac{2z + 2}{5}T2 + \frac{2z + 5}{9}T3 + \frac{2z + 8}{13}T4 + \frac{2z + 11}{17}T5 +$ &c.

The ſum of 9 of the firſt terms is 1.2157.0599.7306.1360.6 ; and to obtain the ſum of the reſt, put 10 for z, and the tenth term for T, and by computation you will obtain

T$=$

T =.0050.1271.8406.8834.46	.0952.4164.9730.7854.7
T2= - - - 1833.9213.6837.20	8069.2540.2083.7
T3= - - - - - 15.5605.4494.38	43.2237.3595.5
T4= - - - - - - 2425.8219.16	5224.8472.0
T5= - - - - - - - - 56.8742.44	103.7118.6
T6= - - - - - - - - - 1.7922.51	2.9017.4
T7= - - - - - - - - - - 707.95	1047.8
T8= - - - - - - - - - - - 33.45	46.1
T9= - - - - - - - - - - - - 1.83	2.3

$$S=.0953.2277.9839.9238.1$$

which being added to S, the sum of the first, you will have
1.3110.2877.7146.0598.7 for the value of the series, that is, for the
length of the *elastic* curve, when extended into a right line ; and this
number *Bernouilli* hath determined to consist between the limits 1.308
and 1.315. And if its ordinate be added to the length of the elastic curve,
we shall have this number 1.9100.9889.4513.8559.8, which is the semi-
periphery of an ellipsis having 1 and $\sqrt{2}$ for its axes. And I hope these
examples are sufficient, for I would not dwell upon series which may be
summed up accurately by this proposition.

S C H O L I U M.

THIS theorem exhibits the areas of binomial curves whose ordinates are
contained under this form $x^\theta \times \overline{e + f x^n}|^\lambda$ with ease, yet in this case only,
when $e + f x^n = 0$, or $x^n = -\dfrac{e}{f}$, that is, when the series converges very
slow for the area. But when the areas are not to be produced be-
yond 8 or 9 figures, it is sufficient to find the sum of four of the
first terms, for S will give the sum of the rest with little trouble ; nay if
no initial terms be got, but the transformation be begun at the first term,
the value of S will approximate fast enough to the value of the whole se-
ries. But the series in the theorem is rendered more general, and is ex-
tended to cases which don't belong to quadratures, as follows :

Let the equation to the series be $T^1 = \dfrac{zz + m}{zz + nz + r} T$; and let us put

$a = n$

$$a = n - \frac{r-m}{n}, \qquad\qquad T_2 = \frac{T}{n-1} \times \frac{ma + \overline{n-a} \times r}{zz + nz + r},$$

$$b = n + 2 - \frac{r-m+n+1}{n+2}, \qquad T_3 = \frac{T_2}{n+1} \times \frac{mb + \overline{n-b+2} \times \overline{r+n+1}}{zz + \overline{n+2}.z + \overline{r+n+1}},$$

$$c = n + 4 - \frac{r-m+2n+4}{n+4}, \qquad T_4 = \frac{T_3}{n+3} \times \frac{mc + \overline{n-c+4} \times \overline{r+2n+4}}{zz + \overline{n+4}.z + \overline{r+2n+4}},$$

$$d = n + 6 - \frac{r-m+3n+9}{n+6}, \qquad T_5 = \frac{T_4}{n+5} \times \frac{md + \overline{n-d+6} \times r + 3n+9}{zz + \overline{n+6}.z + \overline{r+3n+9}},$$

$$e = n + 8 - \frac{r-m+4n+16}{n+8}, \qquad T_6 = \frac{T_5}{n+7} \times \frac{me + \overline{n-e+8} \times r + 4n+16}{zz + \overline{n+8}.z + \overline{r+4n+16}},$$

&c.   &c.

and S will be =

$$\frac{z+a-1}{n-1}T + \frac{z+b-1}{n+1}T_2 + \frac{z+c-1}{n+3}T_3 + \frac{z+d-1}{n+5}T_4 + \frac{z+e-1}{n+7}T_5 + \&c.$$

E   X   A   M   P   L   E.

Let there be propofed this feries $\frac{1}{4} + \frac{1.1}{2.4}A + \frac{3.3}{4.6}B + \frac{5.5}{6.8}C + \frac{7.7}{8.10}D +$

$\frac{9.9}{10.12}E$, &c. which is defined by the equation $T^{I} = \frac{z}{z+\frac{1}{2}} \times \frac{z}{z+\frac{3}{2}}T$, $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$, &c. being the fucceffive values of the abfciffa z. This equation cannot

be compared with $T^{I} = \frac{z-m}{z} \times \frac{z-n}{z-n+1}T$, namely with that in the pro-

pofition, in which however there are two factors $z-n$, and $z-n+1$ dif-

fering by unity, whereof one is in the numerator, and the other the de-

nominator ; fince the differences of the factors in the numerator and de-

nominator are $\frac{1}{2}$ and $\frac{3}{2}$, in the equation defining the propofed feries.

Therefore I multiply the factors into one another, and there comes out T^{I}

$= \frac{zz}{zz + 2z + \frac{3}{4}}T$, and I proceed to the equation in the *Scholium* ; which

compared with the other, I have $m = 0$, $n = 2$, and $r = \frac{3}{4}$; and thefe being

fubftituted, there arifes

$a =$

$$a = \tfrac{11}{4}, \qquad T2 = \tfrac{1}{4} \times \frac{9}{4zz+8z+3} T,$$

$$b = \tfrac{49}{16}, \qquad T3 = \tfrac{3}{16} \times \frac{25}{4zz+16z+15} T2,$$

$$c = \tfrac{100}{44}, \qquad T4 = \tfrac{5}{14} \times \frac{49}{4zz+24z+35} T3,$$

$$d = \tfrac{181}{32}, \qquad T5 = \tfrac{7}{32} \times \frac{81}{4zz+32z+63} T4,$$

$$e = \tfrac{301}{46}, \qquad T6 = \tfrac{9}{46} \times \frac{121}{4zz+40z+99} T5,$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

$$\text{and } S = \frac{8z+5}{1.8} T + \frac{16z+33}{3.16} T2 + \frac{24z+85}{5.24} T3 + \frac{32z+161}{7.32} T4 + \text{\&c.}$$

Now seek the sum of six of the first terms, and there will come out
.6106.6818.2373.0. Then substitute the seventh term for T, and $\tfrac{11}{7}$ the
seventh value for z, and by calculation you will find

T = .0036.3492.9656.98		.0258.9887.3806.0
T2 = ... 1825.5785.11		5210.5053.3
T3 = ... 29.7131.92		59.6739.9
T4 = ... 8425.62		1.3879.7
T5 = ... 339.30		491.0
T6 = ... 17.50		23.1
T7 = ... 1.09		1.3
		S = .0259.5158.9994.3

S being now found, let it be added to the sum of the first terms, and
there will come out .6366.1977.2367.3 for the value of the proposed se-
ries ; and if unity be divided by it, the quotient will give the area of a circle.
And thus are series transformed, which are defined by an equation
of this sort $T^1 = \dfrac{z^3 + az^2 + bz + c}{z^3 + dz^2 + ez + f} T$, or by one more general. And we may

observe, that the equation $T^1 = \dfrac{zz + m}{zz + nz + r} T$ in the scholium, is not ex-
tended therefore to fewer cases, because a term is wanting in the numera-
tor, in which z is of one Dimension ; for that may always be taken away
by changing the beginning of the abscissa, and by that means the theorem
is rendered more simple.

P

P R O P O S I T I O N XII.

R *Equired to transform a series defined by an equation of this form*
$$T \times z^{\gamma} + a z^{\theta-1} + b z^{\theta-2} + \&c. + r T^{I} \times z^{\theta} + c z^{\theta-1} + d z^{\theta-2} + \&c. = 0,$$
into another converging very swift.

Assume $m = c - \dfrac{b-d}{a-c} - \dfrac{1}{r+1}$, $n = m + \dfrac{a-c}{r+1}$; and by the tenth propofi-tion $\dfrac{r}{r+1} \times \dfrac{z+m}{z+n} T$ will be equal to the feries nearly. Therefore let that quantity be the firft term of the transformed feries ; and to find the fecond, put $S = \dfrac{r}{r+1} \times \dfrac{z+m}{z+n} T + S_2$; then by the ninth propofition feek the equation to the terms of the feries S_2, and out of that let there be taken a quantity that is nearly equal to S_2, the fame way, as the approximation was before found to the feries S, and that quantity will be the fecond term of the transformed feries ; and by proceeding thus, you may find as many following terms as you pleafe. $2. E. I.$

E X A M P L E I.

Suppose this feries $\sqrt{12}$ into $1 - \dfrac{1}{3\cdot3} + \dfrac{1}{5\cdot9} - \dfrac{1}{7\cdot27} + \dfrac{1}{9\cdot81} - \dfrac{1}{11\cdot243} +$ &c. were to be transformed, which Dr *Halley* gave for the quadrature of the circle. The relation of the terms is defined by the equation $zT + 3T^{I}\overline{z+1} = 0$, in which the values of z are $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. and by comparing this with the equation in the problem, $r = 3$, $a = 0$, $b = 0$, $c = 1$, $d = 0$, and from thence $m = \frac{3}{4}$, $n = \frac{5}{4}$, which being wrote we fhall have $\frac{1}{4} \times \dfrac{4z+3}{4z+2} T$ for the firft term of the feries transformed.

Suppose $S = \frac{1}{4} \times \dfrac{4z+3}{4z+2} T + S_2$; then by writing the fucceeding values of the variable quantities for the preceding, there will come out $S - T = \frac{1}{4} \times \dfrac{4z+7}{4z+6} T^{I} + S_2 - T_2$, the difference of which equations will give $T = \frac{1}{4} \times \dfrac{4z+3}{4z+2} T - \frac{1}{4} \times \dfrac{4z+7}{4z+6} T^{I} + T_2$, and from thence $T_2 = \frac{1}{4} \times \dfrac{4z-1}{4z+2} T + \frac{1}{4} \times \dfrac{4z+7}{4z+6} T^{I}$. But by the equation to the feries, $T^{I} = -\frac{1}{3} T \times \dfrac{z}{z+1}$; which

being

being fubftituted, you will have $T2 = -\frac{1}{7} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T$, in which writing the latter variable quantities for the former, it will become $T^{I}2 = -\frac{1}{7} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{3}{2z+5} T^{I}$, or by fubftituting $-\frac{1}{7}T \times \frac{z}{z+1}$ for T^{I}, there will come out $T^{I}2 = +\frac{1}{4} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{1}{2z+5} \times \frac{z}{z+1} T$; by this value of $T^{I}2$, T is $= \frac{1}{2} \times \overline{2z+2} \times \overline{2z+3} \times \overline{2z+4} \times \overline{2z+5} T^{I}2$; and for the value of $T2$ before found, $T = -\frac{4}{3} \times \overline{2z+1} \times \overline{2z+2} \times \overline{2z+3} T2$. Laftly, by making thefe two values of T equal to one another, we fhall have $\overline{zz + \frac{1}{2}z} \times T2 + 3T^{I}2 \times \overline{zz + \frac{9}{2}z + 5} = 0$, an equation alfo to the terms of the feries $S2$, which compared with that in the propofition, gives $r = 3$, $a = \frac{1}{4}$, $b = 0$, $c = \frac{9}{2}$, $d = 5$; hence $m = 3$, $n = 2$; and therefore $\frac{3}{4} \times \frac{z+3}{z+2} T2$ will be equal to the feries $S2$ nearly, and is confequently the fecond term of the transformed feries.

To find the third affume $S2 = \frac{3}{4} \times \frac{z+3}{z+2} T2 + S3$; by the differential method $S2 - T2$ will be $= \frac{3}{4} \times \frac{z+4}{z+3} T^{I}2 + S3 - T3$, which fubtracted from the former, there remains $T2 = \frac{3}{4} \times \frac{z+3}{z+2} T2 - \frac{3}{4} \times \frac{z+4}{z+3} T^{I}2 + T3$, from which there arifes $T3 = \frac{1}{4} \times \frac{z-1}{z+2} T2 + \frac{3}{4} \times \frac{z+4}{z+3} T^{I}2$. But by the equation to the feries $S2$, $T^{I}2 = -\frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$, which being fubftituted in the value of $T3$, we have $T3 = -\frac{1}{4} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T2$; where by writing the confequent values of the variable quantities for the preceeding, there will arife $T^{I}3 = -\frac{1}{4} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} T^{I}2$; or by fubftituting for $T^{I}2$ its value, it will be $T^{I}3 = \frac{1}{4} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} \times \frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$. If by the values of the terms $T3$, and $T^{I}3$, $T2$ be got, there will refult $\overline{zz + \frac{1}{2}z} T3 + 3T^{I}3 \times \overline{zz + \frac{11}{2}z + 14} = 0$, which is an equation to the terms of the feries $S3$, and comparing it with that in the theorem, it is $a = \frac{1}{4}$, $b = 0$, $c = \frac{11}{2}$, $d = 14$; confequently $m = \frac{11}{4}$, $n =$

$\frac{11}{4}$, $n = \frac{14}{4}$, and therefore $\frac{1}{4} \times \frac{4z+21}{4z+14} T_3$ will be the approximation to the series S_3, or the third term of the transformed series.

And by a like procefs, by putting $S_3 = \frac{1}{4} \times \frac{4z+21}{4z+14} T_3 + S_4$, you will find the fourth term to be $\frac{1}{4} \times \frac{4z+30}{4z+20} T_4$, T_4 being $= -\frac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T_3$. And in like manner the transformed series will be produced at pleafure; but the progreffion of the terms now appears thus, being

$$T_2 = -\tfrac{1}{8} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T,$$

$$T_3 = -\tfrac{1}{8} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T_2,$$

$$T_4 = -\tfrac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T_3,$$

$$T_5 = -\tfrac{1}{8} \times \frac{10}{2z+10} \times \frac{11}{2z+11} \times \frac{12}{2z+12} T_4,$$

&c.

And $S = \frac{1}{4}$ into $\frac{4z+3}{4z+2} T + \frac{4z+12}{3z+8} T_2 + \frac{4z+21}{4z+14} T_3 + \frac{4z+30}{4z+20} T_4 +$ &c.

The fum of the firft ten terms is $3.1415.9051.0938.0800.9964.2$; the eleventh fubftituted for T, and $\frac{11}{4}$ for z, give

$$
\begin{array}{ll}
+T = .0000.0279.3565.0014.1347.8 & \\
-T_2 = \text{-------} 172.5274.8297.5 & \\
+T_3 = \text{----------} 1474.5938.7 & \\
-T_4 = \text{-------------} 3.8136.0 & \\
+T_5 = \text{---------------} 192.2 & \\
-T_6 = \text{----------------} 1.5 &
\end{array}
$$

.0000.0214.2791.3363.1147.5	.0000.0000.0139.7472.6106.4
1244.1885.8	3.3215.2
171.7	1.4
$+.0000.0214.2791.4607.3205.0$	$-.0000.0000.0139.7475.9323.0$

Now by taking the negative from the affirmative, there will remain $.0000.0214.2651.7131.3882.0$, which added to the fum of the firft terms makes

makes 3.1415.9265.3589.7932.3846.2, for the value of the proposed feries. And these six terms of the feries transformed perform the same thing as thirty two of the simple feries. But the advantage of these theorems will more evidently appear in feries converging very flow, to whose values we cannot come by a mere addition of terms.

E X A M P L E II.

LET this feries $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} +$ &c. be proposed to be transformed; whose equation is $zz \overline{T} + zz + 2z + 1 \, T^I = 0$; 1, 2, 3, 4, 5, &c. being the values of z succeeding in order. Here we shall have $a = 0$, $b = 0$, $c = 2$, $d = 1$, $r = 1$; whence $m = 1$, $n = 0$, and therefore $\frac{1}{4} \times \frac{z+1}{z} T$ is the first term of the transformed feries.

To find the second, suppose $S = \frac{1}{4} \times \frac{z+1}{z} T + S2$; and by the differential method there will come out $S - T = \frac{1}{4} \times \frac{z+2}{z+1} T^I + S2 - T2$, which deducted from the former equation, leaves $T = \frac{1}{4} \times \frac{z+1}{z} T - \frac{1}{4} \times \frac{z+2}{z+1} T^I + T2$; from which we shall find $T2 = \frac{1}{4} \times \frac{z-1}{z} T + \frac{1}{4} \times \frac{z+2}{z+1} T^I$; but the equation to the feries S gives $T^I = \frac{-zzT}{zz+2z+1}$, which substituted, there will come out $T2 = -\frac{2z+1}{2z} \times \frac{T}{z+1|^I}$. Now write the consequent values of the variable quantities for the preceding, and there will arise $T^I2 = -\frac{2z+3}{2z+2} \times \frac{T^I}{z+2|^I}$; or again, by putting for T^I its value, $T^I2 = \frac{1}{4} \times \frac{2z+3}{z+1|^I} \times \frac{zz}{z+2|^I} T$. By the values of the terms T2 and T^I2 exterminate T, and you will have $\overline{z^4 + \frac{1}{4} z^I} \times T2 + \overline{z^4 + \frac{11}{4} z^I + 15 z^2 + 14z + 4} \times T^I2$, which is the equation for the terms of the feries S2; and comparing it with that in the proposition, gives $a = \frac{1}{4}$, $b = 0$, $c = \frac{11}{4}$, $d = 15$, $r = 1$; and therefore $m = 3$, $n = \frac{1}{4}$. Wherefore $\frac{z+3}{2z+1} T2$ is the value of the feries S2 *quamproxime*, or the second term of the feries transformed; and so by proceeding as in the above example, you will find

$$T2 = -\frac{2z+1}{2z} \times \frac{T}{z+1|^I}, \quad T3 = -\frac{2z+2}{2z+1} \times \frac{8T2}{z+2|^I}, \quad T4 = -\frac{2z+3}{2z+2} \times \frac{27T3}{z+3|^I},$$

&c.

Q And

And S$=\dfrac{z+1}{2z}$T$+\dfrac{z+3}{2z+1}$T2$+\dfrac{z+5}{2z+2}$T3$+\dfrac{z+7}{2z+3}$T4$+$ &c.

Or, for an easier computation, put

A$=$T, B$=\dfrac{A}{\overline{z+1}|^3}$, C$=\dfrac{8B}{\overline{z+2}|^3}$, D$=\dfrac{27C}{\overline{z+3}|^3}$, E$=\dfrac{64D}{\overline{z+4}|^3}$, &c.   And

S will be $=\dfrac{1}{2z}$ into $\overline{z+1}$A$-\overline{z+3}$B$+\overline{z+5}$C$-\overline{z+7}$D$+\overline{z+9}$E$-$ &c.

The fum of ten terms of the feries to be fummed up, under their proper figns, is .8179.6217.5610.9851.3.   Then to get the fum of the reft, fubftitute the eleventh term, that is, $\frac{1}{11}$ for T, and 11 for the refpective value of z, and you will have

A$=\frac{1}{11}$, B$=\dfrac{1.1.1}{12.12.12}$A, C$=\dfrac{2.2.2}{13.13.13}$B, D$=\dfrac{3.3.3}{14.14.14}$C, &c.

And S$=\frac{1}{11}$ into 6A$-$7B$+$8C$-$9D$+$10E$-$ &c.

The computation is as follows :

A $=$.0082.6446.2809.9173.5
B $= - - - -$ 478.2675.2372.2
C $= - - - - - -$ 1.7415.2944.5
D $= - - - - - - - -$ 171.3604.0
E $= - - - - - - - - -$ 3.2495.0
F $= - - - - - - - - - -$ 991.7
G $= - - - - - - - - - - -$ 43.6
H $= - - - - - - - - - - - -$ 2.6
I $= - - - - - - - - - - - - -$ 2

Then

.0495.8677.6859.5041.0   .0000.3347.8726.6605.4
13.9322.3556.0   1542.2436.0
32.4950.0   1.0908.7
523.2   33.8
2.8   ——————————————
—————————————— $-$.0000.3348.0296.9983.9
$+$.0495.8691.6214.4073.0

The difference of thefe fums, divided by 11, gives S $=$ .0045.0485.7813.1280.8; which added to the fum of the firft ten, makes .8224.6703.3424.1132.1, for the value of the feries; and when this number is half of that found in the firft example of the eleventh propofition, we may conclude that both methods of computation were rightly made.   For the feries 1$-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+$ &c. when the terms are alternately

nately negative and affirmative, is half of $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ &c. in which all have the same sign.

E X A M P L E III.

LET this series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$ &c. be proposed, which is defined by the equation $zT + \overline{z+1}T' = 0$, in which the values of the abscissa are $\frac{1}{1}, \frac{3}{1}, \frac{5}{1}, \frac{7}{1}$, &c. and, by making a computation according to this proposition, we shall come to the following rule. Put $A = T$, then

$$B = \frac{1}{2}A \times \frac{1}{2z+1} \times \frac{1}{\overline{z+1}|^2},$$

$$C = \frac{1}{2}B \times \frac{2}{2z+3} \times \frac{9}{\overline{z+2}|^2},$$

$$D = \frac{1}{2}C \times \frac{3}{2z+5} \times \frac{25}{\overline{z+3}|^2},$$

$$E = \frac{1}{2}D \times \frac{4}{2z+7} \times \frac{49}{\overline{z+4}|^2},$$

&c.

And S will be $= \frac{1}{4z}$ into $\overline{2z+1}A - \overline{2z+5}B + \overline{2z+9}C - \overline{2z+13}D + \overline{2z+17}E -$ &c. The sum of twelve terms in the series proposed is .7646.0069.1481.8329.5 ; and writing the thirteenth term $\frac{1}{25} = T$, and $z = \frac{25}{2}$, we shall have

$A = \frac{1}{25}$, $B = \frac{1.1.1}{13.27.27}A$, $C = \frac{2.3.3}{14.29.29}B$, $D = \frac{3.5.5}{15.31.31}C$, $E = \frac{4.7.7}{16.33.33}D$, &c.

And $S = \frac{1}{25}$ into $13A - 15B + 17C - 19D + 21E -$ &c.

```
A = .0400.0000.0000.0000.0
B = - - - - 422.0744.9614.9
C = - - - - - - 6452.6422.0
D = - - - - - - - - 33.5725.4
E = - - - - - - - - - 3776.5
F = - - - - - - - - - - - 73.4
G = - - - - - - - - - - - - 2.2
H = - - - - - - - - - - - - - 1
```

$$
\begin{array}{rr}
.5200.0000.0000.0000.0 & .0000.6331.1174.4223.5 \\
10.9694.9174.0 & .637.8782.6 \\
7.9306.5 & 1688.2 \\
55.0 & 2.7 \\
\hline
+.5200.0010.9702.8535.5 & -.0000.6331.1812.4697.0
\end{array}
$$

Then by dividing the difference of the sums by 25, there comes out S =.0207.9747.1915.6153.5; which, together with the aggregate of the firſt terms, makes .7853.9816.3397.4483.0 for the value of the ſeries to be ſummed.

S C H O L I U M.

As one equation defines an infinite number of ſeries, ſo one transformation ſerves for an infinite number of ſeries; and every example is to be reckoned as a theorem. Thus the transformation in the laſt example ſerves for this general ſeries,

$$
\frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \frac{1}{m+4n} - \&c.
$$

Where it is to be obſerved, that the law for continuing a ſeries tranſformed by this propoſition, does not always anſwer as in the examples we have choſen, but it will in no wiſe incommode the work; ſince after we have collected about ſix terms of the ſeries to be ſummed, three or four of the transformed ſeries will give what is required accurately enough for any uſes whatſoever. For in practice, it is ſeldom neceſſary to continue the computation beyond nine or ten figures; and the thing anſwers the ſame end, whether the terms of the transformed ſeries are affected with the ſame or contrary ſigns, or whether they be aſſignable or not; for the trouble will always be inconſiderable, except when in the equation to the ſeries the quantity r is negative, and at the ſame time nearly equal to unity. Indeed if $r = -\frac{9}{10}$, or $=-\frac{99}{100}$, the computation will then be troubleſome; but theſe caſes the ſkilful analyſt will eaſily avoid, for which therefore it is not worth while to preſcribe a remedy.

Methinks, I might alſo add ſome things concerning ſeries of this ſort,

$$
x + \tfrac{1}{2}x^3 + \tfrac{1}{8}x^5 + \tfrac{1}{14}x^4 + \tfrac{1}{10}x^5 + \&c.
$$
$$
x - \tfrac{1}{6}x^3 + \tfrac{1}{110}x^5 - \tfrac{1}{5040}x^7 + \&c.
$$

Where the terms produced *in infinitum* have not a given ratio to one another, as in ſeries which we have hitherto conſidered; but the preceding are infinitely greater than the conſequent. Theſe ſorts exhibit the number from the given logarithm, or the ſine from the given arc, and they are the moſt ſimple of their kind, which may be transformed by the principles

before laid down. But the thing is difpatch'd more commodioufly without transformations; as in this feries, $x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 +$ &c. which exhibits the number from a given logarithm x, if x be $= 12.3785$, I might reject the characteriftic 12, and feek the number of the logarithm .3785; which would come out in a feries converging faft, becaufe the logarithm now is lefs than unity; and this being given, we may eafily find the number of the logarithm 12.3785; and the thing anfwers the fame end whether the logarithm be tabular or hyperbolic.

And likewife by feeking the fine from a given arc, if it be greater than a quadrant, fubduct it from a femicircle, and there will remain an arc lefs than a quadrant, having the fame fine as the former, as being its fupplement to a femicircle; But an arc lefs than a quadrant will give its fine in a feries converging fwiftly.

Series which are defined by equations that involve three or more terms of a feries, may be fummed accurately, or as near as poffible, from the analyfis above laid down. But it may fuffice to have laid a foundation for fuch like computations, and to have opened a way to others, who have leifure and inclination to purfue this matter further. But left we may feem intirely to have omitted it, we will give a general theorem from the acute M. *De Moivre*'s principles, which is extended as well to fummation as to transformation of feries of this kind.

PROPOSITION XIII.

IN *feries arifing from divifion, there is the fame relation between the terms as the fuccessive fums.*

Suppose this fraction $\dfrac{1}{1 - 3x + xx}$, which refolv'd into a feries is $1 + 3x + 8x^2 + 21x^3 + 55x^4 +$ &c. Then the fuccessive fums will be

$$\frac{1}{1-3x+xx} = 1 + 3x + 8x^2 + 21x^3 + 55x^4 + 144x^5 + \&c.$$

$$\frac{3x - xx}{1-3x+xx} = \quad\ 3x + 8x^2 + 21x^3 + 55x^4 + 144x^5 + \&c.$$

$$\frac{8x^2 - 3x^3}{1-3x+xx} = \qquad\quad 8x^2 + 21x^3 + 55x^4 + 144x^5 + \&c.$$

$$\frac{21x^3 - 8x^4}{1-3x+xx} = \qquad\qquad\quad 21x^3 + 55x^4 + 144x^5 + \&c.$$

$$\frac{55x^4 - 21x^5}{1-3x+xx} = \qquad\qquad\qquad\quad 55x^4 + 144x^5 + \&c.$$

R And

And the meaning of the propofition is, that thefe fums, taken in the fame number, have every where the fame relation as the terms of a feries taken fo many in number. For *Example*, in the prefent cafe, the relation between any three terms fucceeding in order will be $xxT - 3xT^I + T^{II} = 0$; and for that reafon the relation between the fucceffive fums will likewife be $xxS - 3xS^I + S^{II} = 0$, as will be manifeft to any one that will try it. And the propofition is thus demonftrated:

Let r, s, t be given quantities, and affume the equation to the fums $rS + sS^I + tS^{II} = 0$; then by fubftituting the fucceeding values of the variable quantities for the prefent, we fhall have $rS^I + sS^{II} + tS^{III} = 0$, which taken from the former, leaves $rS - rS^I + sS^I - sS^{II} + tS^{II} + tS^{II} - tS^{III} = 0$; in which fubftitute T for $S - S^I$, T^I for $S^I - S^{II}$, and T^{II} for $S^{II} - S^{III}$, and there will come out $rT + sT^I + tT^{II} = 0$; and this is the fame relation as that of the fums affumed at firft. And if there be more or fewer fums, the propofition will be intirely demonftrated the fame way.

C O R O L L A R Y.

HENCE we have a method for fumming up thefe feries from a given relation of terms, as will be made manifeft by the following Examples.

E X A M P L E I.

LET there be given the equation to the terms $rT + sT^I = 0$; and, by this propofition, the relation of the fums will be likewife the fame, as $rS + sS^I = 0$. For S fubftitute its value $S - T$, and there will arife $rS + sS - sT = 0$, whence $S = \frac{s}{r+s}T$; wherefore the fum S is given from its firft term T being given. As if the feries be

$$1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \frac{16}{x^4} + \frac{32}{x^5} + \&c.$$

whofe equation is $2T - xT^I = 0$; then will $r = 2$, $s = -x$; which being wrote, there will come out $S = \frac{-x}{2-x}T$, or $S = \frac{x}{x-2}T$. Now fubftitute any term for T, and $\frac{x}{x-2}T$ will be the fum of it and of all the following terms to *infinitum*. Let T be equal to the firft term, namely unity, and we fhall have $\frac{x}{x-2}$ for the value of the whole feries.

<div align="right">EXAMPLE</div>

E X A M P L E II.

AFTER the same manner, if the equation to three terms be $r\,T + s\,T^I + t\,T^{II} = 0$, the relation of the sums will be $r\,S + s\,S^I + t\,S^{II} = 0$; in which, by writing $S - T$ for S^I, and $S - T - T^I$ for S^{II}, it will become $r\,S + s\,S - s\,T + t\,S - t\,T - t\,T^I = 0$, and from thence $S = \dfrac{s + t\,T + t\,T^I}{r + s + t}$; consequently S is given from two terms being given. Let the series to be summed be

$$1 + 3x + 8x^2 + 21x^3 + 55x^4 + 144x^5 + \&c.$$

in which the relation of the terms is $xx\,T - 3x\,T^I + T^{II} = 0$, hence $r = xx$, $s = -3x$, $t = 1$, which being wrote, we shall have $S = \dfrac{1 - 3x\,T + T^I}{1 - 3x + xx}$; now substitute the first term for T, and the second for T^I, and there will come out $\dfrac{1}{1 - 3x + xx}$ for the value of the series.

Likewise if the equation to the terms be $r\,T + s\,T^I + t\,T^{II} + v\,T^{III} = 0$, S will be $= \dfrac{s + t + v\,T + t + v\,T^I + v\,T^{II}}{r + s + t + v}$; and so on, when the relation is between more terms.

S C H O L I U M.

IT is to be noted, that the relation of the terms, which is variable, approaches nearer to an invariable one, the further the terms are from the beginning; and they become invariable at last at an infinite distance, as in series arising from division. And this I call the *ultimate relation* of terms, to which their relation continually approximates; yet it will never come out accurately such, before the terms be removed an infinite space from the beginning.

But a differential equation defining a series, by rejecting all the powers of the abscissa except the highest, and dividing the remaining equation by it, will give the ultimate relation of the terms. Let this equation $xx\,T \times \overline{zz + 3z} = T^I \times \overline{zz - 5z + 2}$ be proposed, reject all the powers of the abscissa below the square, and there will remain $xx\,Tzz = T^I zz$, which divided by zz, will give $xx\,T = T^I$; and this is the ultimate relation of the terms.

And the ultimate relation, when it is constant, supplies a method for summing up series as near as possible, wherein the relation of terms is variable. If any equation $r\,T \times \overline{zz + az + b} + s\,T^I \times \overline{zz + cz + d} = 0$ be given, the

the ultimate relation of the terms will be $r\mathrm{T}+s\mathrm{T}^{\mathrm{I}}=0$, whence $\mathrm{S}=\frac{s}{r+s}\mathrm{T}$, as near as possible. This equation obtains accurately when the term T is at an infinite distance from the beginning, and nearly when the distance is considerably great. Likewise if the equation be $r\mathrm{T}\times\overline{z+a}+s\mathrm{T}^{\mathrm{I}}\times\overline{z+b}+t\mathrm{T}^{\mathrm{II}}\times\overline{z+c}$, the ultimate relation will be $r\mathrm{T}+s\mathrm{T}^{\mathrm{I}}+t\mathrm{T}^{\mathrm{II}}=0$, and $\mathrm{S}=\frac{s+t\mathrm{T}+t\mathrm{T}^{\mathrm{I}}}{r+s+t}$ nearly;

Consequently by collecting some initial terms, before you begin the computation, the sum of the rest will be had nearly by this method.

From these principles we may likewise correct the approximation continually, as in the following proposition.

PROPOSITION XIV.

E*VERY series* $\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}+$ &c. *in which the ultimate relation of the terms is* $r\mathrm{T}+s\mathrm{T}^{\mathrm{I}}+t\mathrm{T}^{\mathrm{II}}=0$, *by assuming* $n=r+s+t$, *and by putting*

$$\begin{array}{llll}
A_2=rA+sB+tC, & A_3=rA_2+sB_2+tC_2, & A_4=rA_3+sB_3+tC_3,\\
B_2=rB+sC+tD, & B_3=rB_2+sC_2+tD_2, & B_4=rB_3+sC_3+tD_3,\\
C_2=rC+sD+tE, & C_3=rC_2+sD_2+tE_2, & C_4=rC_3+sD_3+tE_3,\\
D_2=rD+sE+tF, & D_3=rD_2+sE_2+tF_2, & D_4=rD_3+sE_3+tF_3,\\
E_2=rE+sF+tG, & E_3=rE_2+sF_2+tG_2, & E_4=rE_3+sF_3+tG_3,\\
\text{&c.} & \text{&c.} & \text{&c.}
\end{array}$$

is divided into the two following parts,

$$\overline{s+t} \; into \; \frac{A}{n}+\frac{A^2}{n^2}+\frac{A^3}{n^3}+\frac{A^4}{n^4}+\frac{A^5}{n^5}+ \text{&c.}$$

$$+t \; into \; \frac{B}{n}+\frac{B_2}{n^2}+\frac{B_3}{n^3}+\frac{B_4}{n^4}+\frac{B_5}{n^5}+ \text{&c.}$$

By this proposition, therefore, a series, in which the ultimate relation of the terms is defined by an equation involving three terms, will be transformed into two here exhibited; whereof the first vanishes, when $s+t=0$. If the relation be only between two terms $r\mathrm{T}+s\mathrm{T}^{\mathrm{I}}=0$, t will be $=0$; and for that reason the latter series will vanish, and that which is proposed to be transformed will go into one only; in which particular case, this proposition will coincide with *M. Monmort's* theorem. If the relation be between more terms than in this proposition, the series to be transformed will run into more series. And in every case the thing will be manifest, from what hath been exhibited; and I would not lose its simplicity, by affecting to comprehend all in too few words.

EXAMPLE

E X A M P L E I.

LET this series $1+4x+9x^2+16x^3+25x^4+36x^5+$ &c. be proposed, where the coefficients are squares of the natural numbers; the differential equation is $xT \times \overline{zz+2z+1} - T^1zz = 0$, and from thence the ultimate relation $xT-T^1=0$, consequently $r=x$, $s=-1$, $t=0$, $n=x-1$; and

$A=1$	$A2=-3x$,	$A3=2x^2$,	$A4=0$,
$B=4x$	$B2=-5x^2$,	$B3=2x^2$,	$B4=0$,
$C=9x^2$	$C2=-7x^3$,	$C3=2x^2$,	&c.
$D=16x^3$	$D2=-9x^4$,	&c.	
$E=25x^4$	&c.		
&c.			

Therefore all the terms after A3 vanish, and by substituting their values, the series transformed, becomes

$$-1 \text{ into } \frac{1}{x-1} - \frac{3x}{\overline{x-1}|^2} + \frac{2xx}{\overline{x-1}|^3};$$

and these three terms added together, make $\dfrac{1+x}{\overline{1-x}|^3}$ for the value of the series; and if we change the sign of x, as well in the series, as in its value, we shall have

$$\frac{1-x}{\overline{1+x}|^3} = 1-4x+9x^2-16x^3+25x^4-36x^5+ \text{ &c.}$$

and from thence likewise

$$\frac{1+6x^2+x^4}{\overline{1-xx}|^3} = 1+9x^2+25x^4+49x^6+81x^8+ \text{ &c.}$$

which may be added, without these difficulties, directly by the proposition.

E X A M P L E II.

LET this series $1+8x+27x^2+64x^3+125x^4+216x^5+$ &c. be proposed; where the coefficients are the cubes of the natural numbers, and the equation to the series is $xT \times \overline{z^3+3z^2+3z+1} - T^1z^3 = 0$; consequently the ultimate relation is $xT-T^1=0$; and from thence $r=x$, $s=-1$, $t=0$, $n=x-1$; and

$A=1$,	$A2=-7x$	$A3=12x^2$,	$A4=-6x^3$,	$A5=0$,
$B=8x$,	$B2=-19x^2$,	$B3=18x^3$,	$B4=-6x^4$,	$B5=0$,
$C=27x^2$,	$C2=-37x^3$,	$C3=24x^4$,	$C4=-6x^5$,	&c.
$D=64x^3$,	$D2=-61x^4$,	$D3=30x^5$,	&c.	
$E=125x^4$,	$E2=-91x^5$,	&c.		
$F=216x^5$,	&c.			
&c.				

Here

Here the terms vanish after A4, and the values substituted in the theorem give

$$-1 \text{ into } \frac{1}{x-1} - \frac{7x}{\overline{x-1}|^2} + \frac{12x^2}{\overline{x-1}|^3} - \frac{6x^3}{\overline{x-1}|^4},$$

and these four terms added together make $\dfrac{1+4x+xx}{\overline{1-x}|^4}$ for the value of the series.

E X A M P L E   III.

Let there be given this series $1-6x+27x^2-104x^3+366x^4-1212x^5$ $+3842x^6-11784x^4+$ &c. to be summed, which is defined by this equation $xxTz+4-2xT^Iz+2-T^{II}z=0$. Now let z be infinitely great, and the ultimate relation will be $xxT-2xT^I-T^{II}=0$; hence $r=xx$, $s=-2x$, $t=-1$, $n=xx-2x-1$; and

A = +1,	A2 = −14x²,	A3 = +29x⁴,	A4 = 0,
B = −6x,	B2 = +44x³,	B3 = −70x⁵,	B4 = 0,
C = +27x²,	C2 = −131x⁴,	C3 = +169x⁶,	&c.
D = −104x³,	D2 = +376x⁵,	D3 = −408x⁷,	
E = +366x⁴,	E2 = −1052x⁶,	&c.	
F = −1212x⁵,	F2 = +2888x⁷,		
G = +3842x⁶,	&c.		
H = −11784x⁷,			
&c.			

Now substitute these values for A, A2, A3, B, B2, B3, and the proposed series will pass into the following finite ones.

$$-\overline{2x+1} \text{ into } \frac{1}{xx-2x-1} - \frac{14x^2}{\overline{xx-2x-1}|^2} + \frac{29x^4}{\overline{xx-2x-1}|^3},$$

$$-1 \text{ into } \frac{-6x}{xx-2x-1} + \frac{44x^3}{\overline{xx-2x-1}|^2} - \frac{70x^5}{\overline{xx-2x-1}|^3};$$

which two added together, make $\dfrac{1}{\overline{1+2x-xx}|^3}$, for the value of the series.   After the same manner the theorem is applied to transformation of series which cannot be summed up; but the demonstration will become manifest from what follows.

Suppose any fraction $\dfrac{a+bx+cx^2+dx^3}{v+tx+sx^2+rx|^n}$, whose numerator is a constant quantity compos'd of any determined number of members, and

let

let its denominator be any power of a quantity, which likewise is composed of any finite number of terms. Then whatever the index n be if the fraction be resolved into a series, there will always be the same ultimate relation of the terms, as if the denominator were the simple power $v + tx + sx^2 + rx^3$, therefore if the series be continually multiplied into this, the product will at last terminate, if n be an integer and affirmative, that is, if the series be summable by a simple equation.

Let this series $1 - 6x + 27x^2 - 104x^3 + 366x^4 - $ &c. be proposed, in which the ultimate relation of the coefficients is $A - 2B - C = 0$; I take the quantity $xx - 2x - 1$, or (by changing the signs) $1 + 2x - xx$, wherein the coefficients are the same as in the ultimate relation, and I conclude the proposed series (provided it be summable) equal to some fraction, whose denominator is some power of the quantity $1 + 2x - xx$. Therefore I put $S = 1 - 6x + 27x^2 - 104x^3 + 366x^4 - $ &c. and I draw each part into $1 + 2x - xx$; and there comes out $S \times \overline{1 + 2x - xx} = 1 - 4x + 14x^2 - 44x^3 + 131x^4 - $ &c. and because it does not yet terminate, I draw it again into the same quantity, and I have $S \times \overline{1 + 2x - xx}|^2 = 1 - 2x + 5x^2 - 12x^3 + 29x^4 - $ &c. then multiplying it a third time, I obtain $S \times \overline{1 + 2x - xx}|^3 = 1$, all the terms vanishing after the first.

Therefore, in short, this proposition amounts to nothing more than to perform a compendious method of multiplication, and likewise the disposition of the terms in transformed series, which makes them converge swiftly, when they do not break off or terminate. Wherefore I leave the demonstration to be investigated by the reader, which is almost the same whether the denominators be trinomials, quadrinomials, &c.

P R O P O S I T I O N XV.

TO *find an equation, whether it be algebraical or fluxional, whose root shall be any given series whatsoever, which is defined by an equation in which the terms of the series are only of one dimension.*

The series is given, from some given initial terms, with a rule for forming the rest, and from these given terms, an equation will be found having that series for the root; which will be understood by the following examples.

E X A M P L E I.

To find an equation whose root is the series $A + Bx + Cx^2 + Dx^3 + Ex^4 + $ &c. wherein the relation of the terms is permanent, namely $rA + sB + tC + vD = 0$. For as much as the relation is invariable, I conclude the series to be equal to a rational fraction, which I thus find out.

Suppose

Suppose y equal to a given series, and by summing the indices of the relation r, s, t, v, in inverse order, I draw y and its value successively into v, tx, sx^2, rx^3, and there will come out

$$vy = vA + vBx + vCx^2 + vDx^3 + vEx^4 + vFx^5 + \&c.$$
$$txy = \quad\;\; tAx + tBx^2 + tCx^3 + tDx^4 + tEx^5 + \&c.$$
$$sx^2y = \qquad\quad\;\; sAx^2 + sBx^3 + sCx^4 + sDx^5 + \&c.$$
$$rx^3y = \qquad\qquad\quad\;\; rAx^3 + rBx^4 + rCx^5 + \&c.$$

But from the supposition $rA + sB + tC + vD = 0$; $rB + sC + tD + vE = 0$, and so *in infinitum*; whence all the terms vanish wherein x is of more than two dimensions, and consequently there will remain

$$vy + txy + sx^2y + rx^3y = vA + \left.\begin{matrix} +vB \\ +tA \end{matrix}\right\} x + \left.\begin{matrix} +vC \\ +tB \\ +sA \end{matrix}\right\} x^2.$$

which is an equation for determining the value of the series. For instance, suppose $A = 2$, $B = -3$, $C = 7$, $r = 3$, $s = -7$, $t = 6$, $v = 4$; and the equation will then become

$$4y + 6xy - 7x^2y + 3x^3y = 8 - 4x^2, \text{ or } y = \frac{8 - 4x^2}{4 + 6x - 7x^2 + 3x^3}.$$

E X A M P L E II.

To find an equation which may have this series $1 - \frac{2}{3}x^2 + \frac{8}{15}x^4 - \frac{16}{35}x^6 + \frac{128}{315}x^8 - \frac{256}{693}x^{10} - \&c.$ for the root, wherein the relations of the coefficients are $2A + 3B = 0$, $4B + 5C = 0$, $6C + 7D = 0$, $8D + 9E = 0$, &c.

Here the equation sought will involve first fluxions, because the numbers in the relations have equal differences. Assume therefore

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \&c.$$
$$x\dot{y} \text{ will be } = 2Bx^2 + 4Cx^4 + 6Dx^6 + 8Ex^8 + \&c.$$
$$\dot{y} + x\dot{y} = A + 3Bx^2 + 5Cx^4 + 7Dx^6 + 9Ex^8 + \&c.$$
$$2x^2\dot{y} + x^2\ddot{y} = 2Ax^2 + 4Bx^4 + 6Cx^6 + 8Dx^8 + \&c.$$

Now add the last equation to the last but one, and there will come out

$$y + x\dot{y} + 2x^2\dot{y} + x^2\ddot{y} = A + \left.\begin{matrix} +2A \\ +3B \end{matrix}\right\} x^2 + \left.\begin{matrix} +4B \\ +5C \end{matrix}\right\} x^4 + \left.\begin{matrix} +6C \\ +7D \end{matrix}\right\} x^6 + \left.\begin{matrix} +8D \\ +9E \end{matrix}\right\} x^8 + \&c.$$

But, from the hypothesis, the relations of the coefficients are $2A + 3B = 0$; $4B + 5C = 0$; $6C + 7D = 0$, &c. and therefore all the terms after A the first vanish, and there remains this finite equation $y + x\dot{y} + 2x^2\dot{y} + x^2\ddot{y} = A = 1$; or $y \times \overline{1 + 2xx} + x\dot{y} \times \overline{1 + xx} = 1$, namely unity being substituted, or the first term of the series, for the coefficient A.

E X A M P L E III.

To find an equation whofe root is the feries $1 - \frac{1}{4}x^2 - \frac{1}{64}x^4 - \frac{1}{336}x^6 -$ $\frac{171}{1044}x^8 -$ &c. where the relations of the coefficients are

$-1.1A - 2.2B = 0$, $1.3B - 4.4C = 0$, $3.5C - 6.6D = 0$, $5.7D - 8.8E = 0$, &c.
or $-A - 4B = 0$, $3B - 16C = 0$, $15C - 36D = 0$, $35D - 64E = 0$, &c.

Here, in this cafe, the defired equation will neceffarily involve fecond fluxions, becaufe the numbers which fhew the relation of the coefficients are of two dimenfions, or products under equidifferent numbers taken two by two. Therefore let

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \&c.$$
$$\dot{y} = 2Bx + 4Cx^3 + 6Dx^5 + 8Ex^7 + \&c.$$
$$x\ddot{y} = 2Bx + 12Cx^3 + 30Dx^5 + 56Ex^7 + \&c.$$
$$\dot{y} + x\ddot{y} = 4Bx + 16Cx^3 + 36Dx^5 + 64Ex^7 + \&c.$$
$$x^2\ddot{y} + x^2\dot{y} - xy = -Ax + 3Bx^3 + 15Cx^5 + 35Dx^7 + \&c.$$

Laftly, by deducting this laft equation from the laft but one, there will remain

$$x\ddot{y} - x^3\ddot{y} + \dot{y} - x^2\dot{y} + xy = \left\{ {+ A \atop + 4B} \right\} x {- 3B \atop + 16C} \} x^3 {-15C \atop +36D} \} x^5 {-35D \atop +64E} \} x^7$$

&c. that is, $x\ddot{y} - x^3\ddot{y} + \dot{y} - x^2\dot{y} + xy = 0$, or $\dot{y} + x\ddot{y} + \dfrac{xy}{1-xx} = 0$; for by reafon of the relation of the coefficients, all the terms on one fide of the equation vanifh.

E X A M P L E IV.

Let an equation be fought, whofe root is $1 + \frac{1}{4}Ax^2 + \frac{9}{16}Bx^4 + \frac{15}{36}Cx^6$ $+ \frac{44}{64}Dx^8 +$ &c. where the relations of the coefficients are $A - 4B = 0$, $9B - 16C = 0$, $25C - 36D = 0$, &c. Suppose

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + Fx^{10} + \&c.\ \text{then will}$$
$$\dot{y} = 2Bx + 4Cx^3 + 6Dx^5 + 8Ex^7 + 10Fx^9 + \&c.$$
$$\text{and } x\ddot{y} = 2Bx + 12Cx^3 + 30Dx^5 + 56Ex^7 + 90Fx^9 + \&c.$$

Then by computation you will find

$$xy + 3x^2\ddot{y} + x^3\ddot{y} = Ax + 9Bx^3 + 25Cx^5 + 49Dx^7 + 81Ex^9 + \&c.$$
$$\dot{y} + x\ddot{y} = 4Bx + 16Cx^3 + 36Dx^5 + 64Ex^7 + 100Fx^9 + \&c.$$

Subduct the latter equation from the former, and you will have

$$xy - \dot{y} \times \overline{1 - 3xx} - x\ddot{y} \times \overline{1 - xx} = \left\{ {+ A \atop -4B} \right\} x {+ 9B \atop -16C} \} x^3 {+25C \atop -36D} \} x^5 {+49D \atop -64E} \} x^7$$
$$+ \&c.$$

T

+ &c. and then, by reason of the relation of the coefficients, the equation will be $xy = \dot{y} \times \overline{1 - 3xx} + x\ddot{y} \times \overline{1 - xx}$.

E X A M P L E V.

REQUIRED that equation, whose root is $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 +$ &c. where the denominators are squares of the natural numbers 1, 2, 3, &c. Assume

$$y = x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 +$$ &c. then will

$$x\dot{y} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 +$$ &c.

whose fluxion is

$$x\ddot{y} + \dot{y} = 1 + x + x^2 + x^3 + x^4 +$$ &c.

that is, $x\ddot{y} + \dot{y} = \dfrac{1}{1 - x}$.

Likewise if the series be $x + \frac{1}{8}x^2 + \frac{1}{27}x^3 + \frac{1}{64}x^4 +$ &c. where the denominators are the cubes of the natural numbers, you will find the equation to be $\dot{y} + 3x\ddot{y} + x^2\dddot{y} = \dfrac{1}{1-x}$; and if the series be $x + \frac{1}{16}x^2 + \frac{1}{81}x^3 + \frac{1}{256}x^4$, the denominators being biquadrates, the equation will come out $\dot{y} + 6x\ddot{y} + 7x^2\dddot{y} + x^3\ddddot{y} = \dfrac{1}{1-x}$, and so on.

S C H O L I U M.

FROM a given relation therefore of the coefficients, series are reduced either to fractions or to fluxions, and with as much ease in all cases, whether the series be determined by the relation of more or fewer terms. For by taking the fluxion of the series, terms are drawn into the indices of the powers, which are always equidifferent, and again into other quantities equidifferent, by taking the second fluxions; and so on, one may in this manner form the relations of the terms, whatsoever they are, by pursuing the methods now laid down.

But to a compleat summation, fluxions must be reduced to fluents; and when it cannot be done, we must conclude, that the series which we have been considering is not of their number which can be summed. But it conduces much to the extending this doctrine, that summations be reduced to fluxions, because the method for finding fluents, tho' imperfect, yet is better known, and more cultivated, than the method made use of for finding the sums from the differences. But however they mutually help each other; for if fluents can be found, they will exhibit the sums; and, vice versâ, sums found will give the fluents.

From the propositions already delivered, and others easily to be deduced
from

from those principles, we may find the root of any equation whatsoever very accurately in numbers, if it can be reduced into a simple series, tho' converging very slow. But if the series neither approximates quickly, nor the relation of the terms be simple and proper, which is defined by a differential equation, there will be found some difficulty. For if in the equation to be resolved there be found its root, or the fluxions thereof, of more than of one dimension ; or if these be multiplied into one another, and such terms cannot be exterminated ; I say, in those cases, the root is not reducible to a simple series composed of the powers of the abscissa, at least by any method hitherto known to me. But the powers of the abscissa, howsoever high they are in the equation to be resolved, by no means hinder the simplicity of the series.

But, upon this occasion, I will endeavour to remove that difficulty hitherto unattempted, which yet is the greatest in the reduction of fluxional equations. Therefore we must know that those fluxional equations which involve all the conditions of the problem, and consequently determine all the coefficients in their roots, may be resolved in the same manner as adfected literal equations by *Newton*'s rules. Nevertheless it often happens, that the equations do not determine all the coefficients of the terms in their roots ; because the constant quantities vanish, and are quite lost out of the equation, in proceeding from the fluents to the fluxions ; but they vanish several ways, according as the fluxion is taken, which we will here show by an example.

Let there be proposed this equation $y^2 = a^2 + bx + \dfrac{x^4}{c^2}$; x flowing uniformly, and writing unity for its fluxion, we shall have, by the direct method of fluxions, $2y\dot{y} = b + \dfrac{4x^3}{c^2}$, whose root, extracted according to Sir *Isaac Newton*'s method, will be the same as the root of the fluent $y^2 = bx + \dfrac{x^4}{c^2}$; because the quantity aa is lost, and in this case the first term of the series is not determined, which depends on the vanished quantity aa.

Likewise if the proposed equation be first divided by x, there will come out $\dfrac{y^2}{x} = \dfrac{aa}{x} + b + \dfrac{x^3}{c^2}$, whose fluxion is $\dfrac{2y\dot{y}}{x} - \dfrac{y^2}{x^2} = \dfrac{-aa}{x^2} + \dfrac{3x^2}{c^2}$, or by drawing it into x^2, is $2xyy - y^2 = -aa + \dfrac{3x^4}{cc}$. And its root taken purely is the same as that of the fluent, $y^2 = a^2 + \dfrac{x^4}{cc}$; in which case the second term of the series is not determined, which depends on the coefficient b, now lost.

Lastly,

Laſtly, if the equation be divided by x^4, we ſhall have $\frac{y^2}{x^4}=\frac{aa}{x^4}+\frac{b}{x^3}+$ $\frac{1}{cc}$; and from thence the fluxion will be $\frac{2y\dot{y}}{x^4}-\frac{4y^2}{x^5}=\frac{-4aa}{x^5}-\frac{3b}{x^4}$, or $2xy\dot{y}$ $-4y^2=-4aa-3bx$, which reſolved by the common rules, will give the ſame root as the fluent $y^2=a^2+bx$; conſequently the fifth term of the ſeries, which depends on the coefficient cc, is not determined. Hence it appears that the fluxional equations

$$2y\dot{y}=b+\frac{4x^3}{c^3},$$

$$2xy\dot{y}-y^2=-aa+\frac{3x^4}{cc},$$

$$2xy\dot{y}-4y^2=-4aa-3bx.$$

proceed from one fluent only, $y^2=aa+bx+\frac{x^4}{cc}$. By the firſt equation the firſt term of the ſeries is not determined; by the ſecond equation the firſt, but not the ſecond term is determined, by the third the four firſt terms of the ſeries are determined, but the fifth left undetermined; therefore diverſe fluxional equations may be produced from the ſame fluent. But the method of reſolution will not be perfect, till we are able to exhibit all the roots of the different fluents from which any propoſed fluxion can any way be produced; for it is neceſſary to have the ratio of the quantities, which either have actually vaniſhed, or may vaniſh.

For we cannot extract the root from a fluxional equation, as if no quantity had vaniſhed, and then add a given quantity to the fluent found, as in the quadrature of curves; therefore the indeterminate terms are very often invariable quantities, and the addition of a given quantity to the root found is not equivalent to the addition of a quantity to the equation to be reſolved. Thus the ſame fluxion $2y\dot{y}=b+\frac{4x^3}{cc}$ may come from either of the fluents $yy=aa+bx+\frac{x^4}{cc}$, or $yy=bx+\frac{x^4}{cc}$; wherefore the roots obtain quite different forms, namely the firſt $A+Bx+Cx^2+$ &c. the latter $Ax^{\frac{1}{2}}+Bx^{\frac{3}{2}}+Cx^{\frac{5}{2}}+$ &c.

Moreover a fluxional equation may be made to involve all the coefficients which the fluent involves, and yet for all that, all the terms cannot ſtill be determined; as will appear by the following example. The equation $y^2=aa+bx+\frac{x^4}{cc}$ exhibits directly the fluxion $2y\dot{y}=b+\frac{4x^3}{cc}$, which if

drawn

drawn into y will become $2y^2\dot{y}=by+\dfrac{4x^3y}{cc}$; and then substituting for yy,

its value $aa+bx+\dfrac{x^4}{cc}$, we shall have at last $2aa\dot{y}+2bx\dot{y}+\dfrac{2x^4\dot{y}}{cc}=by+$

$\dfrac{4x^3y}{cc}$; and this is the fluxional equation involving all the coefficients which its fluent involves, wherein nevertheless the first term of the series is not determined; and this will always happen, unless there be a term in the equation, which neither involves the root, nor any fluxion there.

There are many more difficulties not of less moment, which any one will easily perceive to be increased in the fluxions of inferior orders; for in second fluxions, two terms no where depending on one another may be indeterminate, and three in third fluxions, and so on. But it often happens that all the terms may be determined even from an equation involving fluxions of inferior orders; and all that has been said of a fluxion whose fluent is known, is equally true of fluxions whose fluents are known.

The root of any equation whatsoever is a quantity, which being wrote for the letter denoting that root, will make all the terms of the resulting equation vanish. But terms can only vanish two ways, either by the contrariety of the signs in homologous terms, or when a constant quantity is found in the fluent, for there is left no sign or remains of it in the fluxion. As if $y=Ax^n$; then $\dot{y}=nAx^{n-1}$, $\ddot{y}=\overline{nn-n}Ax^{n-2}$; if $n=0$, the value of the first fluxion will vanish; if $nn-n=0$, that is, $n=0$, or $n=1$, the value of the second fluxion will vanish, and that without any homologous terms which may destroy them. And these are the principles whereby all difficulties must be resolved which occur in the resolution of fluxional equations.

Let this equation $r^2\dot{y}^2=r^2\dot{x}^2-\dot{x}^2y^2$ be proposed; or, putting unity for $\dot{x}$, $r^2\dot{y}^2=r^2-y^2$. By *Newton's* parallelogram, or his other methods, you will find the root

$$y=x-\dfrac{x^3}{6r^2}+\dfrac{x^5}{120r^4}-\dfrac{x^7}{5040r^6}+\ \&c.$$

For the square thereof wrote for y^2, and the square of the fluxion for $\dot{y}^2$, will make all the terms of the resulting equation destroy themselves, by a contrariety of signs; but let us see whether there may not be another root which cannot be found by this way. For that end, suppose Ax^n to be the first term of a series; or $y=Ax^n$ nearly; then will $\dot{y}=nAx^{n-1}$, and therefore $\dot{y}^2=n^2A^2x^{2n-2}$, and $yy=A^2x^{2n}$; which being wrote, there results $n^2r^2A^2x^{2n-2}=r^2-A^2x^{2n}$, or, by drawing the whole into x^2,

$$n^2r^2A^2x^{2n}=r^2x^2-A^2x^{2n+2};$$

U

where

where it appears that the term $n^2 r^2 A^2 x^{2n}$ vanishes when $n=0$; therefore substitute o for n, and the equation will become

$$0 \times r^2 A^2 x^0 = r^2 x^2 - A^2 x^2.$$

Therefore in this case, $r^2 A^2 x^0$ vanishes wherein x is of the least dimensions; and for this reason it will be $A x^0$, or the constant quantity A, the first term of the series converging the swifter by how much x is less; and by pursuing this computation you will find, by the common methods,

$$y = A \times \overline{1 - \frac{x^2}{2 r^2} + \frac{x^4}{24 r^4} - \frac{x^6}{720 r^6}} + \&c.$$

But the quantity A is not determined from a fluxional equation. But if, we take the fluxion of the equation $r^2 \dot{y}^2 = r^2 - y^2$, we shall have $2 r^2 \dot{y} \ddot{y} = -2 y \dot{y}$, and by dividing and transposing, $r^2 \ddot{y} + y = 0$. Now let us put $y = A x^n$, then will $\dot{y} = n A x^{n-1}$, and $\ddot{y} = \overline{nn - n} A x^{n-2}$, which being substituted there comes out $\overline{nn - n} \times r^2 A x^{n-2} + A x^n = 0$; where it appears that the index $n-2$ cannot be compared with the index n; and consequently no root can be found by that method. Nevertheless, by putting the coefficient $nn - n = 0$, we shall have $n = 0$, and $n = 1$; consequently A, or A x, may be the first term of a series, as we have found above.

The first series gives the sine, and the latter the cosine from a given arc x; and the coefficient A in the latter is equal to the radius r. For if you put y for the sine, or cosine, you will always light upon this equation, $r^2 \dot{y}^2 = r^2 - y^2$; which is explicable by no root, except the two already mentioned. And from this example, we may observe the form of the series always to be determined from the equation, tho' the coefficients can't be determined; and when the coefficient of any term is not determined, the index of x therein will always be found by putting some member of the resulting equation equal to nothing. But when the coefficient of a term is determined, the index of x in the same is found by a comparison of two indices in the resulting equation, and that by *Newton*'s Rules.

Let the equation $\ddot{y} + a^2 y - x \dot{y} = x^2 \ddot{y} = 0$, be given to be resolved, where x flows uniformly, and unity is wrote for its fluxion. Suppose $y = A x^n$ nearly, and $\dot{y}$ will be $= n A x^{n-1}$, $\ddot{y} = \overline{nn - n} A x^{n-2}$, which substituted in the equation there results $\overline{nn - n} A x^{n-2} + \overline{a a - nn} A x^n = 0$; now put the coefficient $nn - n = 0$, and there will come out $n = 0$, or $n = 1$; these values substituted for n in the equation, exhibit

$$0 \times A x^{-2} + a a A x^0 = 0, \text{ and}$$
$$0 \times A x^{-1} + \overline{a a - 1} A x = 0;$$

and

and when in either cafe a term vaniſhes wherein x is of the leaſt dimenſi-ons, I conclude o, or 1, is the index of x in the firſt term of the series, which converges ſo much the faſter, by how much x is leſs; confequent-ly we may aſſume

$$y = A + Bx^2 + Cx^4 + Dx^6 + \&c.$$
$$\text{or } y = Ax + Bx^3 + Cx^5 + Dx^7 \&c.$$

In the refulting equation $\overline{nn - n}Ax^{n-2} + \overline{aa - nn}Ax^n = 0$; now put the coefficient $aa - nn = 0$, and n will be $= \pm a$; therefore write $+a$, and $-a$ for n, and there will come out

$$\overline{aa - a}Ax^{a-2} + 0 \times Ax^a = 0.$$
$$\overline{aa + a}Ax^{-a-2} + 0 \times Ax^{-a} = 0.$$

Therefore in each cafe there the term vaniſhes wherein x is of the greateſt dimenſions, and for that reafon, $+a$, or $-a$, will be the index of the firſt term of the series converging ſo much the faſter, by how much x is greater: For we may aſſume

$$y = Ax^a + Bx^{a-2} + Cx^{a-4} + Dx^{a-6} + \&c.$$
$$\text{or } y = Ax^{-a} + Bx^{-a-2} + Cx^{-a-4} + Dx^{-a-6} + \&c.$$

The determinate coefficients of thefe four series gives the following values of the root;

$$A + \frac{0 - aa}{1.2}Ax^2 + \frac{4 - aa}{3.4}Bx^2 + \frac{16 - aa}{5.6}Cx^2 + \frac{36 - aa}{7.8}Dx^2 + \&c.$$

$$Ax + \frac{1 - aa}{2.3}Ax^2 + \frac{9 - aa}{4.5}Bx^2 + \frac{25 - aa}{6.7}Cx^2 + \frac{49 - aa}{8.9}Dx^2 + \&c.$$

$$Ax^a - \frac{\overline{a.a - 1}}{4.a - 1} \times \frac{A}{x^2} \quad \frac{\overline{a - 2.a - 3}}{8.a - 2} \times \frac{B}{x^2} - \frac{\overline{a - 4.a - 5}}{12.a - 3} \times \frac{C}{x^2} \quad \frac{\overline{a - 6.a - 7}}{16.a - 4} \times \frac{D}{x^2} - \&c.$$

$$\frac{A}{x^a} - \frac{\overline{a.a + 1}}{4.a + 1} \times \frac{A}{x^2} \quad \frac{\overline{a + 2.a + 3}}{8.a + 2} \times \frac{B}{x^2} - \frac{\overline{a + 4.a + 5}}{12.a + 3} \times \frac{C}{x^2} \quad \frac{\overline{a + 6.a + 7}}{16.a + 4} \times \frac{D}{x^2} - \&c.$$

The two firſt belong to the multiplication or diviſion of a circular arc; but the laſt to the area of the hyperbola. And, I hope, thefe examples are fufficient for illuftrating the rule here laid down for determining the in-dex; it is deduced from the fifth propoſition; and by it, and *Newton's* rule, the extraction of roots in infinite ſeries is brought to perfection, as any one will eafily perceive, who is moderately verfed in thofe matters, which authors have hitherto advanc'd upon this ſubject. For it will be eafily demonſtrated that an equation does not admit of a ſeries for the root, which is not found by this method; but I don't here ſpeak of ſeries which confiſts of terms wherein the indeterminate quantity x hath indeterminate indices.

O F

OF THE

INTERPOLATION of SERIES.

PART II.

LET there be any right line PQ given in pofition, upon which let there be erected as many ordinates A, B, C, D, &c. as you pleafe, parallel to one another, and equally diftant from each other ; and and let thefe ordinates denote the terms of a regular feries, continually increafing or decreafing, and affected with the fame fign ; and one and the fame curve will pafs through the extremities of all, which will be deter-

A a P B b C c D d E e F f Q G g
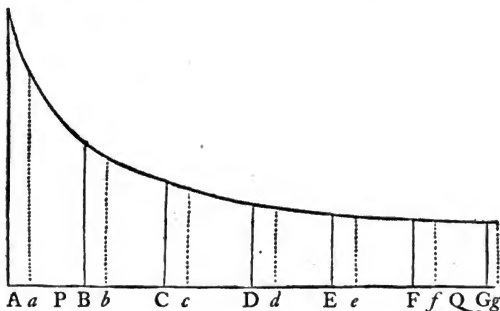
mined from the given equation to the feries, that is, from the given equation expreffing generally the relation between two or any more fucceffive ordinates.

If from a given differential equation and algebraic equation of this curve can be inveftigated, being that which defines the relation between the abfciffas and correfponding ordinates ; every ordinate will be had from its given abfciffa by the refolution of adfected equations ; and confequently an abfolute interpolation of a feries, being that which confifts in the affignation of any primary or intermediate term from its given place in the fe-
ries ;

ries; but when the algebraic equation of a curve cannot be found, which often is the case, nothing further is to be expected than to exhibit the value of any term sought by a converging series, or perhaps by the quadrature of curves.

But I here speak of equidistant terms, whose relations are produced by writing the equidifferent numbers successively for the abscissa z in the differential equation defining the series; for a common interval of the ordinates standing upon the abscissa, is proportional to the invariable increment of the indeterminate quantity z.

P R O P O S I T I O N XVI.

I*F the common distance of the primary terms and of the intermediate ones be the same, and there be given one of the intermediate terms, all will be given from the given equation to the primary.*

In the above figure, let A, B, C, D, &c. denote the primary terms, and *a, b, c, d,* &c. the intermediate; and let the distances of the primary terms AB, BC, CD, &c. be equal to the distances of the intermediate terms, *a b, b c, c d,* &c. each to each; I say, that all the intermediate terms are given, from any one of them being given, and also the differential equation defining the relation of the primary terms.

For an equation which assigns the relation of the primary terms, defines a curve passing thro' their extremities; and the equation which defines the relation of the intermediate terms, defines likewise a curve passing thro' their extremities. But by the definition of primary and intermediate terms, the same curve passes thro' the extremities of both; wherefore an equation which defines the curve, will define the relation of the terms in both series. But the equation is given from the hypothesis, consequently the law for continuing the intermediate terms is given, wherefore all the terms will be given from any one being given. $\mathcal{Q}. E. D.$

C O R O L L A R Y.

If in any differential equation expressing the relation of the primary terms A, B, C, D, &c. the successive values of the abscissa z be $-PA$, PB, PC, PD, &c. any point P being taken for the beginning of the abscissa; then will be had the relations of the intermediate terms by writing for z successively $-Pa$, P*b*, P*c*, P*d*, &c. in the same equation; for the differential equation generally expresses the relation between any two ordinates placed at a certain distance from one another, whether they fall in the series of primary or of intermediate ordinates.

X   EXAMPLE

EXAMPLE I.

In the geometrical progreſſion of the terms 1, x, x^2, x^3, x^4, &c. if the terms ſtanding in the middle between the primary one be a, b, c, d, e, &c. then will $b=ax$, $c=bx$, $d=ex$, $e=dx$, &c. whereof the relation is the ſame as of the primary terms.

EXAMPLE II.

If the primary terms be 1, 1, 2, 6, 24, 120, 720, &c. whoſe relations are B=A, C=2B, D=3C, E=4D, &c. and the term directly ſtanding in the middle between the two firſt terms 1 and 1, be a, the reſt will be given by theſe equations, $b=\frac{1}{2}a$, $c=\frac{3}{2}b$, $d=\frac{5}{2}c$, $e=\frac{7}{2}d$, &c. as theſe ſtand in the midſt between any two primary terms, being hence equally diſtant from them.

But if a denote the terms between the firſt and ſecond, whoſe diſtance from the firſt is a third part of the common diſtance, put $b=\frac{4}{3}a$, $c=\frac{7}{3}b$, $d=\frac{10}{3}c$, $e=\frac{13}{3}d$, &c. and a, b, c, d, e, &c. will be the intermediate terms, whereof every one is diſtant a third part of the common interval from the primary one going before it, or two third parts of the ſame interval from the primary one immediately following it. It appears therefore that the relation of the intermediate terms is given from the interpolation of the relation of the primary terms; and it is always interpolable, becauſe it is aſſigned by the equation to the ſeries.

EXAMPLE III.

Let this ſeries 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{11}$, &c. be propoſed, which is produc'd by a continual multiplication of the numbers $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{6}$, $\frac{4}{7}$, &c. and let a be the term in the middle between the firſt and ſecond, and put $b=\frac{3}{7}a$, $c=\frac{5}{9}b$, $d=\frac{7}{11}c$, $e=\frac{9}{13}d$, &c. then b, c, d, e, &c. will be the remaining intermediate terms which ſtand in the middle between any two primary terms.

SCHOLIUM.

If an equation to the ſeries involve three terms, two muſt be given, and if four, three muſt be given, and ſo on, that the other intermediate terms may be had. And of this kind is the ſeventh propoſition of *Newton*, in his Quadrature of curves; which not only anſwers in curves, but alſo in any ſeries. And this theorem is uſeful as often as the intermediate term is ſought, which ſtands near the beginning of the ſeries; for in that caſe its value will come out in a ſeries converging very ſlow. Wherefore, firſt, a reſpective intermediate term muſt be ſought, that is ſufficiently remote from the beginning, whoſe value will come out in a ſeries converging ſwiftly;

swiftly; then from this given term, we muſt go back to that firſt propoſed, by the relation of terms, as in this propoſition.

P R O P O S I T I O N   XVII.

*E*VERY *ſeries is interpolable, whoſe terms conſiſt of interpolable factors.*

Let $A \times a \times \alpha$, $B \times b \times \beta$, $C \times c \times \gamma$, $D \times d \times \delta$, &c. be a ſeries whoſe terms conſiſt of three factors; I ſay, that it is interpolable, if the three ſeries of factors be interpolable, *viz.* A, B, C, D, &c. *a*, *b*, *c*, *d*, &c. and α, β, γ, δ, &c.

For becauſe the intermediate terms in the compoſed ſeries are formed the ſame way from the correſponding intermediate terms in the ſimple ſeries, as the primary terms from the primary terms, they will be found by drawing the reſpective intermediate terms of the ſimple ſeries into one another. As if T be a term between B and C, *t* the reſpective term between *b* and *c*, and τ the reſpective term between β and γ, then the correſponding term in the compoſed ſeries, *viz.* that between $B \times b \times \beta$, and $C \times c \times \gamma$, will be the product under theſe three, namely $T \times t \times \tau$. And if there be more or fewer factors, the propoſition will likewiſe be demonſtrated. *2. E. D.*

C O R O L L A R Y.

Hence in a ſeries to be interpolated, if two or more ſeries of factors are accurately interpolable, they may be rejected from the computation; and then the reſt are to be interpolated by methods afterward to be laid down. For interpolation is not raſhly to be undertaken; but before the beginning of the work, we muſt enquire what ſeries is the moſt ſimple, on whoſe intercalation depends that of the propoſed ſeries. And this preparation is for the moſt part intirely neceſſary, in order to come at neat and elegant concluſions.

E X A M P L E   I.

If the ſeries to be intercalated be $1, \frac{1}{6}x, \frac{1}{40}x^2, \frac{1}{112}x^3, \frac{1}{112}x^4$, &c. firſt I reſolve it into three ſeries of ſimple factors after the following manner.

$$x^0, \quad x^1, \quad x^2, \quad x^3, \quad x^4, \text{ &c.}$$
$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \tfrac{1}{5}, \text{ &c.}$$
$$1, \quad \tfrac{1}{3}, \quad \tfrac{1}{5}, \quad \tfrac{7}{7}, \quad \tfrac{15}{126}, \text{ &c.}$$

and it is manifeſt that the two firſt are interpolable, not the third; which, ſince it is more ſimple, will yet be more eaſily interpolated, than that propoſed. As ſuppoſe that be required which is in the middle between the ſecond and third of a compounded ſeries; it appears that the correſpondent

terms

terms in the firſt and ſecond of the ſimple ſeries is $x^{\frac{1}{2}}$ and $\frac{1}{4}$ reſpec-
tively; in the third let the correſpondent term be called T, then the pro-
duct of theſe three $x^{\frac{1}{2}}$, $\frac{1}{4}$, and T, or $\frac{1}{4}Tx\sqrt{x}$, will be that which is de-
ſired.

Conſequently therefore the interpolation of a compounded ſeries is re-
duc'd to the interpolation of a more ſimple one.

E X A M P L E II.

If a ſeries of this form 1, $\frac{r}{p}A$, $\frac{r+1}{p+1}B$, $\frac{r+2}{p+2}C$, $\frac{r+3}{p+3}D$, &c. be given,
we can interpolate the ſeries of the numerators and denominators apart,
that is, the ſeries

$$1, r, r.\overline{r+1}, r.\overline{r+1}.\overline{r+2}, \text{ &c.}$$
$$1, p, p.\overline{p+1}, p.\overline{p+1}.\overline{p+2}, \text{ &c.}$$

Then any term in the ſeries of the numerators, divided by the reſpec-
tive term in that of the denominators, will give the correſpondent term in
the propoſed ſeries.   If the difference between r and p be a ſmall number,
there is no need of this artifice; but when the aforeſaid difference is great,
it is neceſſary to interpolate the numerators and denominators apart.

S C H O L I U M.

To this may likewiſe many ſuch like preparations be referred; for ex-
ample, in this ſeries,
&c. $e, d, c, b, a,$ A, B, C, D, E, &c. running on hence *in infinitum*
there be deſired the term in the middle between the two middle primary
terms a and A.   Draw the primary terms which are equally diſtant from
the middle term into one another; that is, A into a, B into b, &c. and
there will be compoſed a new ſeries.
&c. Dd, Cc, Bb, Aa, Aa, Bb, Cc, Dd, &c. running on both ways
in infinitum, whoſe terms equally diſtant from the middle are equal to one
another, and whoſe term in the middle, between Aa and Aa, will be the
ſquare of the term ſought, which ſtands in the middle between a and A, in
the ſeries before propoſed.   Therefore the term ſought may be found by
the interpolation of either of the two ſeries.

It is to be obſerved that the deſired term may ſtand in divers ſeries, and
from that conſideration is ſometimes eaſier to be found.   As if any term
ſought ſhould lie in the midſt between the firſt and ſecond, in each of the
following ſeries,

$$1, r, r.r+1, r.r+1.r+2, \text{ &c.}$$
$$1, \frac{1}{p}, \frac{1}{p.p+1}, \frac{1}{p.p+1.p+2}, \text{ &c.}$$

Then

Then by drawing the respective terms into one another, there will be produced a new series.

$$1, \frac{r}{p}A, \frac{r+1}{p+1}B, \frac{r+2}{p+2}C, \frac{r+3}{p+3}D, \&c.$$

between whose first and second term, that which possesses the middle place is equal to the square of that first proposed.

Interpolation will sometimes succeed well by logarithms, especially if the differences of the terms be very great; but these and such like things are to be learned more by experience than any other way. For as in common algebra, the whole art of the analyst does not consist in the resolution of adfected equations, but in bringing of problems thereto; so likewise in this analysis, there is less dexterity required for the resolution of differential equations, or interpolation of series; for in finding series which determine the unknown quantities, and which are fit for an interpolation, there is far greater difficulty.

PROPOSITION XVIII.

IF the terms of two series be formed by multiplying into one another continually the fractions whose numerators and denominators increase by a perpetual addition of unity, and if there be the same numerators in each; I say, that the term of one, whose distance from the beginning is the difference of the factors in the other, is equal to the term of this whose distance from the beginning is the difference of the factors in the other series, if so be that the first terms be equal to one another.

Suppose two series whose first terms A and a are equal to one another,

$$B = \frac{r}{p}A, \quad C = \frac{r+1}{p+1}B, \quad D = \frac{r+2}{p+2}C, \quad E = \frac{r+3}{p+5}D, \&c.$$

$$b = \frac{r}{q}a, \quad c = \frac{r+1}{q+1}b, \quad d = \frac{r+2}{q+2}c, \quad e = \frac{r+3}{q+3}d, \&c.$$

in which the numerators are the same; I say, that the term of the former whose distance from the beginning is equal to the difference of the factors in the latter, namely $q-r$, is equal to the term of the latter, whose distance from the beginning is $p-r$, namely, equal to the difference of the factors in the first series; and you must note, when $p-r$, or $q-r$ is a negative quantity, that the terms here meant, stand at these distances before the first.

Let us assume any term of the first series, for instance the seventh, putting A = 1, viz.

Y

$$G = 1 \times \frac{r}{p} \times \frac{r+1}{p+1} \times \frac{r+2}{p+2} \times \frac{r+3}{p+3} \times \frac{r+4}{p+4} \times \frac{r+5}{p+5}.$$

And firſt, if $p-r=0$, or $p=r$; then $p+1$ will be $=r+1$, $p+2=r+2$, &c. therefore all the numerators and denominators mutually deſtroy one another, and there remains $G=1$.

If $p-r=1$, p will be $=r+1$; and from thence $p+1=r+2$, $p+2=r+3$, &c. in which caſe all the numerators vaniſh except the firſt, and all the denominators except the laſt; G being $=1 \times \frac{r}{p+5}$, or $G = 1 \times \frac{r}{r+6}$, becauſe $p+5$ is equal to $r+6$.

If $p-r=2$, or $p=r+2$; then $p+1=r+3$, $p+2=r+4$, &c. and now all the numerators will vaniſh except the two firſt, and all the denominators except the two laſt; G being $=1 \times \frac{r}{p+4} \times \frac{r+1}{p+5}$, or $G = 1 \times \frac{r}{r+6} \times \frac{r+1}{r+7}$, becauſe $p+4=r+6$, and $p+5=r+7$.

And likewiſe if $p-r=3$, or $p=r+3$, all the numerators will vaniſh except the three firſt, and all the denominators except the three laſt; and in this caſe G will be $=1 \times \frac{r}{r+7} \times \frac{r+1}{r+7} \times \frac{r+2}{r+8}$. And univerſally in the value of G, there will always be ſo many numerators and denominators as there are units in $p-r$, as in the following table.

$$p-r=0, \quad G=1,$$
$$p-r=1, \quad G=1 \times \frac{6}{r+6},$$
$$p-r=2, \quad G=1 \times \frac{r}{r+6} \times \frac{r+1}{r+7},$$
$$p-r=3, \quad G=1 \times \frac{r}{r+6} \times \frac{r+1}{r+7} \times \frac{r+2}{r+8},$$
$$p-r=4, \quad G=1 \times \frac{r}{r+6} \times \frac{r+1}{r+7} \times \frac{r+2}{r+8} \times \frac{r+3}{r+9},$$
$$\text{&c.}$$

Therefore if we put $q=r+6$, or $q-r=6$, the term of this ſeries $1, \frac{r}{p}A, \frac{r+1}{p+1}B$, &c. whoſe interval from the beginning is $q-r$, or 6, is equal to the term of this ſeries $1, \frac{r}{q}a, \frac{r+1}{q+1}b$, &c. whoſe interval

from

from the beginning is $p-r$. And in the fame way, the propofition will be manifeft in other cafes. *2. E. D.*

C O R O L L A R Y.

Hence if the difference of the factors p and r be not very great, the term of the former feries, how far foever remote from the beginning, will always be determined by a term in the latter feries, which is near the beginning, as will appear from the following examples.

E X A M P L E I.

If $r=3$, $p=5$, $q=10$; and thefe values are fubftituted, the two feries will become

$$1, \frac{3}{5}, \frac{3\cdot4}{5\cdot6}, \frac{3\cdot4\cdot5}{5\cdot6\cdot7}, \frac{3\cdot4\cdot5\cdot6}{5\cdot6\cdot7\cdot8}, \&c.$$

$$1, \frac{3}{10}, \frac{3\cdot4}{10\cdot11}, \frac{3\cdot4\cdot5}{10\cdot11\cdot12}, \frac{3\cdot4\cdot5\cdot6}{10\cdot11\cdot12\cdot13}, \&c.$$

But $q-r=7$, $p-r=2$; therefore the term in the firft feries, whofe interval from the beginning is 7, will be equal to that term in the latter, whofe diftance from the beginning is 2; or, which is all the fame, the eighth term $\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}{5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11}$ of the firft feries is equal to $\frac{3\cdot4}{10\cdot11}$, the third term of the laft feries. And it is to be obferved when the difference between p and r is a whole number, then every term of the firft feries is always equal to fome primary one in the latter.

E X A M P L E II.

Let the firft feries be $1, \frac{3}{1}A, \frac{4}{3}B, \frac{5}{5}C, \frac{6}{7}D$, &c. and becaufe the factors increafe by 2's, divide the numerators and denominators by two, and there will come out this feries $1, \frac{\frac{1}{2}}{\frac{1}{2}}A, \frac{2}{\frac{1}{2}+1}B, \frac{3}{\frac{1}{2}+2}C$, &c. where now the increafe of the factors is unity; and confequently this feries may be compared with that in the theorem, r coming out $=1$, and $p=\frac{1}{2}$. Suppofe likewife m to be the diftance between the firft term of the feries and any other, and m will be $=q-r$, or $m=q-1$, and $q=m+1$; which being fubftituted, the latter feries will become

$$1, \frac{a}{m+1}, \frac{2b}{m+2}, \frac{3c}{m+3}, \frac{4d}{m+4}, \&c.$$

in which, that term whofe diftance from the beginning is $p-r$, or $-\frac{1}{2}$,

will

will be equal to that term of the first series, whose interval from the beginning is any quantity m; that is, the term of the first series, how far soever placed from the beginning, namely, at however great a distance m, will always be equal to that term of the latter, which stands before the first in the middle of the common distance.

Or if there be taken this series $1, \frac{1}{2}A, \frac{1}{3}B, \frac{1}{4}C, \frac{1}{5}D$, &c. whose terms are the reciprocal of those of which we have now treated, r will be $=\frac{1}{2}$, $p=1$; and if m be the interval between the first term and any other, m will be $=q-r$, or $m=q-\frac{1}{2}$, and $q=m+\frac{1}{2}$, and the latter series will become

$$1, \quad \frac{a}{2m+1}, \quad \frac{3b}{2m+3}, \quad \frac{5c}{2m+5}, \quad \frac{7d}{2m+7}, \text{ &c.}$$

in which the term at the distance $p-r$, or $\frac{1}{2}$ from the beginning, that is, the term in the midst between the first and second, will be equal to any term whatsoever of the first series whose distance from the beginning is any quantity m. As if the *one thousandth* and *first* term of a series be desired, whereof the interval from the beginning is a *thousand*, m will be $=$ 1000, and the term of the series

$$1, \quad \frac{a}{2001}, \quad \frac{3b}{2003}, \quad \frac{5c}{2005}, \quad \frac{7d}{2007}, \text{ &c.}$$

which stands in the middle between the first 1 and second $\frac{1}{1001}$, will be equal to the *one thousandth* and *first* term of this series $1, \frac{1}{2}A, \frac{1}{3}B, \frac{1}{4}C, \frac{1}{5}D$, &c. whose intermediate terms are likewise found the same way; for if for m be wrote $999\frac{1}{2}$, the series will come out

$$1, \quad \frac{a}{2000}, \quad \frac{3b}{2002}, \quad \frac{5c}{2004}, \quad \frac{7d}{2006}, \text{ &c.}$$

whose term in the middle between the first and second, is equal to that term of this series $1, \frac{1}{2}A, \frac{1}{3}B, \frac{1}{4}C, \frac{1}{5}D$, &c. which stands in the middle between the *thousandth* and *thousandth* and *first*.

EXAMPLE III.

If a term of this series $1, \frac{1}{4}A, \frac{1}{7}B, \frac{1}{10}C, \frac{1}{13}D$, &c. whose interval from the beginning suppose any quantity m; first divide the numerators and denominators by their common increment 3, and the series will become $1, \frac{\frac{1}{3}}{\frac{4}{3}}A, \frac{\frac{1}{3}}{\frac{7}{3}}B$, &c. Therefore r will be $=\frac{1}{3}$, $p\frac{1}{3}$, $q-r=q-\frac{1}{3}=$ m; and thence $q=m+\frac{1}{3}$, whence the last series will become

$$1, \quad 2a$$

$$1, \quad \frac{a}{m+1}, \quad \frac{2b}{m+2}, \quad \frac{3c}{m+3}, \quad \frac{4d}{m+4}, \quad \&c.$$

in which, that term whose distance from the beginning is $p-r$, or $-\frac{1}{2}$, will be equal to that term of the first series, whose interval from the beginning is any quantity m; that is, the term of the first series, how far soever placed from the beginning, namely at however great a distance m, will always be equal to that term of the latter, which stands before the first in the middle of the common distance.

Or if there be taken this series $1, \frac{1}{2}A, \frac{1}{4}B, \frac{1}{6}C, \frac{1}{7}D$, &c. whose terms are the reciprocal of those of which we have now treated, r will be $=\frac{1}{2}$, $p=1$; and if m be the interval between the first term and any other, m will be $=q-r$, or $m=q-\frac{1}{2}$, and $q=m+\frac{1}{2}$, and the latter series will become

$$1, \quad \frac{a}{2m+1}, \quad \frac{3b}{2m+3}, \quad \frac{5c}{2m+5}, \quad \frac{7d}{2m+7}, \quad \&c.$$

in which the term at the distance $p-r$, or $\frac{1}{2}$ from the beginning, that is, the term in the midst between the first and second, will be equal to any term whatsoever of the first series whose distance from the beginning is any quantity m. As if the *one thousandth* and *first* term of a series be desired, whereof the interval from the beginning is a *thousand*, m will be $=$ 1000, and the term of the series

$$1, \quad \frac{a}{2001}, \quad \frac{3b}{2003}, \quad \frac{5c}{2005}, \quad \frac{7d}{2007}, \quad \&c.$$

which stands in the middle between the first 1 and second $\frac{1}{1001}$, will be equal to the *one thousandth* and *first* term of this series $1, \frac{1}{2}A, \frac{1}{4}B, \frac{1}{6}C, \frac{1}{7}D$, &c. whose intermediate terms are likewise found the same way; for if for m be wrote $999\frac{1}{2}$, the series will come out

$$1, \quad \frac{a}{2000}, \quad \frac{3b}{2002}, \quad \frac{5c}{2004}, \quad \frac{7d}{2006}, \quad \&c.$$

whose term in the middle between the first and second, is equal to that term of this series $1, \frac{1}{2}A, \frac{1}{4}B, \frac{1}{6}C, \frac{1}{7}D$, &c. which stands in the middle between the *thousandth* and *thousandth* and *first*.

E X A M P L E   III.

IF a term of this series $1, \frac{4}{7}A, \frac{4}{7}B, \frac{4}{7}C, \frac{4}{10}D$, &c. whose interval from the beginning suppose any quantity m; first divide the numerators and denominators by their common increment 3, and the series will become $1, \frac{1}{4}A,$

$\frac{1}{\frac{1}{3}}A$, $\frac{1}{\frac{1}{3}}B$, &c.   Therefore r will be $= \frac{1}{3}$, $p = \frac{1}{3}$, $q - r = q - \frac{1}{3} = m$; and thence $q = m + \frac{1}{3}$, whence the laſt ſeries will become

$$1, \quad \frac{2a}{3m+2}, \quad \frac{5b}{3m+5}, \quad \frac{8c}{3m+8}, \quad \frac{11d}{3m+11}, \&c.$$

whoſe term at the diſtance from the beginning $p - r$, or $-\frac{1}{3}$, that is, the term which ſtands at a diſtance before the firſt, equal to the third part of the common interval, is equal to that term of the former ſeries, which is diſtant from the beginning by the diſtance m, however great.

Of the Differences of QUANTITIES.

SUPPOSE a, b, c, d, e be a ſeries of any number of quantities; and if the firſt be taken from the latter, there will remain the firſt differences, $b - a$, $c - b$, $d - c$, $e - d$; then if the firſt of theſe differences be likewiſe taken from the laſt, there will remain the ſecond differences $c - 2b + a$, $d - 2c + b$, $e - 2d + c$; the differences of which again make the third differences $d - 3c + 3b - a$, $e - 3d + 3c - b$, of the quantities a, b, c, d, e. And thus we might proceed till we come to the ultimate difference, as in the following table.

Suppoſe $1 - x$ be a binomial, in which the unciæ $+1$, -1 are the ſame as the coefficients in the firſt differences; then the unciæ of the ſquare $1 - 2x + x^2$, namely $+1$, -2, $+1$, will be the coefficients in the ſecond differences; likewiſe the unciæ of the cube 1, -3, $+3$, -1, will be the coefficients in the third differences. And, in general, the coefficients in any order of the differences will be the unciæ in the corresponding power of the binomial. And theſe being premiſed, we may go at once to any order of differences, without conſidering the intermediate quantities.

Two quantities have a firſt difference, three have a ſecond, four a third; neither can they have any further. But it ſometimes happens that a certain order of differences conſtitutes a progreſſion of equal differences, in which caſe, no further differences can be had, how great ſoever the number of the quantities be. Thus, in arithmetical progreſſion, the firſt differences

differences are equal, consequently the second are not given. And in a series of squares 1, 4, 9, 16, 25, &c. whose roots are equidifferent, the first differences 3, 5, 7, 9, &c. are in arithmetical progression, the second are equal, and for that reason the third are nothing. So likewise in a series of cubes 1, 8, 27, 64, 125, 216, &c. the first differences are 7, 19, 37, 61, 91, &c. the second 12, 18, 24, 30, &c. the third 6, 6, 6, &c. therefore equal, consequently the fourth nothing at all.

And, universally, suppose A, B, C, D, E be any given quantities whatsoever, but z a variable quantity; then in the expression $A + Bz + Cz^2 + Dz^3 + Ez^4$ write any equidifferent numbers successively for z, and the last differences of the quantities coming out will be determined by the highest power z^4, without any regard to the lower ones. So in this case the fourth difference is the last, because the fourth power z^4 is here the highest.

The differences very often constitute a converging series in cases when they do not terminate. As if a, b, c, d, e, &c. were nearly equal amongst themselves, and their first differences $b-a$, $c-b$, $d-c$, $e-d$, &c. were likewise nearly equal to one another, and likewise the second and the following differences were very nearly equal to one another; then will a, $b-a$, $c-2b+a$, $d-3c+3b-a$, &c. constitute a converging series. Likewise the differences of the terms whose relation is defined by this equation

$$T \times \overline{z^\theta + az^{\theta-1} + bz^{\theta-2} +} \ \&c. = T' \times \overline{z^\theta + cz^{\theta-1} + dz^{\theta-2} +} \ \&c.$$

taken after the foregoing manner, will make a converging series. But we must not expect that the differences of any quantities either converge or terminate; this happens only in those quantities which increase or decrease accurately, or very nearly, with that celerity as some certain powers of equidifferent numbers.

Of the Description of Curves through given Points.

Sir *Isaac Newton*, in his last letter to Mr *Oldenburgh*, dated *October* 24, 1676, to be communicated to M. *Leibnitz*, after he had shewn an expedient for avoiding series too much complex, in the quadrature of trinomial curves, says, " But these I reckon of lesser moment, that when simple
" series are not sufficiently manageable, I have another method, not yet
" communicated, whereby we approach to the thing sought at pleasure ;
" its foundation is a commodious, easy, general solution of this pro-
" blem : *To describe a geometrical curve, which shall pass through any*
" *number of given points.* *Euclid* hath taught how to describe a circle
" through three points ; so likewise may a conic section be described thro'
" five

" five given points ; and a curve of three dimensions through seven given
" points (so that I have in my power the description of all curves of this
" order, which are determined only by seven points.) These are done
" geometrically, without any calculation. But the above problem is of
" another kind ; and tho' at first look it seems intractable, yet the thing
" is otherwise, for it is one of the most elegant and neatest sort that I had
" a desire to solve."

Newton, in the 60th proposition of his *Universal Arithmetic,* teaches
how to describe a conic section through four points ; or rather shews a
method for finding an equation to the parabola, which shall pass through
four given points.   And by the same method we can describe a line of the
third order through nine points, and a line of the fourth order through
fourteen ; and so in others.   But our design does not require a solution so
general ; for it is sufficient to describe a parabolic figure through the ex-
tremities of as many ordinates as you please, which are parallel to one an-
other, and likewise to the axis of the curve ; but neither does an organic
description of a curve by the motion of angles, or any other way, con-
duce any thing to the present purpose, the thing being the same whether
the curve be actually described, or be conceived to be so. For curves
here are in no wise necessary, but only so far as they are a help to the un-
derstanding, in conceiving the problem rightly. For a description of the
parabola through given points, is altogether the same problem as the as-
signation of quantities from their given differences, which is always accom-
plish'd by algebra only, and that by the resolution of simple equations.

PROPOSITION XIX.

*SUPPOSE there be given a series of equidistant ordinates, proceeding
only one way* in infinitum, *and it be required to find a parabolic curve
which shall pass through the extremities of them all.*

Let A, A₁, A₂, A₃, A₄, &c. denote the equidistant ordinates stand-
ing perpendicularly upon the abscissa. Collect their first differences B,
B₁, B₂, B₃, &c. their second C, C₁, C₂, &c. their third D, D₁, &c.
and so on ; so that A be the first ordinate, B the first difference of the
two first ordinates, C the second difference of the three first ordinates. D
the third difference of the four first, &c. But the differences must be
collected by taking the former every where from the latter ; that is, by
putting B=A₁−A, B₁=A₂−A₁, &c. C=B₁−B, &c. and so in the
rest, by putting those negative, which arise from subduction of a greater
quantity from a lesser.
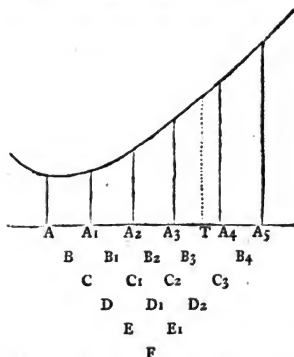
Thefe being premis'd, let T be any ordinate in general, primary or intermediate, whofe diftance from the firft ordinate A, namely, let AT be to the common interval of equidiftant ordinates, as the indeterminate quantity z is to unity; and it will be

$$T = A +$$
$$B \times \frac{z}{1} +$$
$$C \times \frac{z}{1} \times \frac{z-1}{2} +$$
$$D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} +$$
$$E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} +$$
$$F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} +$$
$$\&c.$$

This is the value of every ordinate T lying the fame way from the firft ordinate A, as the reft; but if it lie to the other hand, the fign of the abfciffa z muft be changed. For I put the abfciffa affirmative which goes from the beginning towards the ordinates on the right, but negative if it go the contrary way. And the propofition is thus demonftrated.

Conceive the ordinate T to be carried by a parallel motion along the

A a   abfciffa,

abſciſſa, ſo that it may come ſucceſſively to the places of the reſt.   And becauſe its diſtance from the firſt ordinate is ſuppoſed to be to the common interval of the ordinates as z to unity, z will ſucceſſively be equal to 0, 1, 2, 3, 4, &c. and in the mean time T will be reſpectively equal to the ordinates A, A1, A2, A3, &c. in their proper place.   Therefore to find the coefficients A, B, C, D, &c. which cauſe the parabola to paſs through the extremities of the ordinates, in the equation to the figure $T = A + B\frac{z}{1} + C\frac{z}{1} \times \frac{z-1}{2} + D\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3}$ + &c. write the ordinates A, A1, A2, A3, &c. ſucceſſively for T; and for z the lengths of the abſciſſa following in order, that is, 0, 1, 2, 3, &c. and there come out theſe equations.

A = A,	whence A = A,
A1 = A + B,	B = A1 − A,
A2 = A + 2B + C,	C = A2 − 2A1 + A,
A3 = A + 3B + 3C + D,	D = A3 − 3A2 + 3A1 − A,
A4 = A + 4B + 6C + 4D + E,	E = A4 − 4A3 + 6A2 − 4A1 + A
&c.	&c.

For from the values of the ordinates A, A1, A2, &c. are the values of the coefficients A, B, C, &c. again found; from which it appears that the firſt ordinate A is the firſt coefficient; alſo the difference of the two firſt ordinates is the ſecond coefficient, and the ſecond difference of the three firſt ordinates is the third coefficient, and ſo *in infinitum*. Therefore the values of the coefficients laid down in the ſolution, cauſe the parabola to paſs through the extremities of the ordinates. *Q. E. D.*

Otherwiſe thus:

Suppoſe in general $T = A + B\frac{z}{1} + C\frac{z}{1} \times \frac{z-1}{1} + D\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3}$ + &c. where A, B, C, D, &c. are the coefficients to be determined; write the conſequent values of the variable quantities T^I, $z+1$ for the antecedent T, z; and there will come out $T^I = A + B\frac{z+1}{1} + C\frac{z+1}{1} \times \frac{z}{2} + D\frac{z+1}{1} \times \frac{z}{2} \times \frac{z-1}{3}$ + &c. from which if the value of T be ſubducted, we ſhall have $T^I - T = B + C\frac{z}{1} + D \times \frac{z}{1} \times \frac{z-1}{2}$ + &c. where, by ſubſtituting T^{II}, T^I, $z+1$, for T^I, T, z, there will come out $T^{II} - T^I = B + C\frac{z+1}{1} + D$ $\frac{z+1}{1}$

$\frac{z+1}{1} \times \frac{z}{2} +$ &c. from which the value of $T^I - T$ subtracted leaves

$$T^{II} - 2T^I + T = C + D\frac{z}{1} + \text{ &c.}$$

And likewise you will find $T^{III} - 3T^{II} + 3T^I - T = D + \text{ &c.}$ Now let T denote the first ordinate, and the corresponding value of the abscissa will be nothing, which being substituted, you will find

$$A = T,$$
$$B = T^I - T,$$
$$C = T^{II} - 2T^I + T,$$
$$D = T^{III} - 3T^{II} + 3T^I - T,$$
$$\text{ &c.}$$

That is, the first coefficient A is equal to the first ordinate T, the second coefficient B is equal to the difference between the two first ordinates T and T', the third C is equal to the second difference of the three first ordinates T, T', T'' the fourth D is equal to the third difference of the four first ordinates; and so in the others, as has been already demonstrated.

E X A M P L E I.

Let there be given five ordinates 1, 4, 2, 3, 9, through whose extremities the parabola must pass. Collect their first differences 3, —2, 1, 6; their second —5, 3, 5, their third 8, 2; and the last —6. Then according to what is laid down, in the solution of the proposition, put the first ordinate, and every first difference respectively, for A, B, C, &c. that is, A=1, B=3, C=—5, D=8, E=—6; but F, G, &c. will be nothing. And these values being substituted, the equation to the parabola comes out

$$T = 1 + 3\frac{z}{1} - 5\frac{z}{1} \times \frac{z-1}{2} + 8\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} - 6\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4},$$

which reduc'd into order, becomes

$$T = \frac{12 + 116z - 111z^2 + 34z^3 - 3z^4}{12}.$$

And to prove the work, write 0, 1, 2, 3, 4, successively for the abscissa z, and instead of T there will come out the five proposed ordinates. But the ordinates may be taken in an inverse order, so that the signs of the

the differences be changed in the alternate orders; then we muſt put A=9, B=−6, C=5, D=−2, E=−6; which being wrote, and the Equation reduc'd into order, we ſhall obtain at laſt T=$\frac{108-92z+2z^2+9z^3+14z^3-3z^4}{12}$, in which if for z be ſubſtituted 0, 1, 2, 3, 4, there will come out the propoſed ordinates in an inverſe order. And here are obtained two equations for the ſame parabola, becauſe the abſciſſa takes its beginning ſometimes from the firſt, and ſometimes from the laſt ordinate.

EXAMPLE II.

REQUIRED now to find an equation to the parabola, which ſhall paſs through the extremities of ſix equidiſtant ordinates, 5, 3, 7, 23, 57, 115; collect their firſt differences, and ſo

5		3		7		23		57		115
	−2		4		16		34		58	
		6		12		18		24		
			6		6		6			

the others, till you come to the laſt, as in the margin; and you will find A=5, B=−2, C=6, D=6, which ſubſtituted, there ariſes

$$T=5-2\frac{z}{1}+6\frac{z}{1}\times\frac{z-1}{2}+6\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3},$$

which reduc'd becomes T=5−3z+z³; and if for z you write ſucceſſively 0, 1, 2, 3, 4, 5, there will come out the ſix propoſed ordinates.

A right line paſſes thro' two points, a conic parabola thro' three, a cubic one thro' four, a biquadratic thro' five; and ſo *in infinitum*. But it ſometimes happens that a curve of an inferior order paſſes thro' more points, as in the laſt example; and the order of the parabola is always denoted by the laſt order of the differences. That if the number of the ordinates be infinite, and a progreſſion of equal differences do not come out; I ſay, in that caſe, the curve will be of infinite dimenſions, the value of T running out into an infinite ſeries.

SCHOLIUM.

IN this ſolution, we have put unity for the common diſtance of the ordinates; but if we had uſed any quantity *n* for the ſame, it would have come out

$$T=A+B\times\frac{z}{n}+C\times\frac{z}{n}\times\frac{z-n}{2n}+D\times\frac{z}{n}\times\frac{z-n}{2n}\times\frac{z-2n}{3n}+ \&c.$$

Suppoſe

Suppose now the 2d ordinate $A_1 = A + \dot{A}n$,

the third $\quad\quad\quad A_2 = A + 2\dot{A}n + \ddot{A}n^2$,

the fourth $\quad\quad\quad A_3 = A + 3\dot{A}n + 3\ddot{A}n^2 + \dddot{A}n^3$,

the fifth $\quad\quad\quad A_4 = A + 4\dot{A}n + 6\ddot{A}n^2 + 4\dddot{A}n^3 + \ddddot{A}n^4$,

&c.

and again you will find $B = \dot{A}n$, $C = \ddot{A}n^2$, $D = \dddot{A}n^3$, $E = \ddddot{A}n^4$, &c. which values being fubftituted for B, C, D, E, &c. there will arife

$$T = A + \dot{A}\frac{z}{1} + \ddot{A}\frac{z}{1} \times \frac{z-n}{2} + \dddot{A}\frac{z}{1} \times \frac{z-n}{2} \times \frac{z-2n}{3} + \text{\&c.}$$

Now let the common interval n be nothing, and $\dot{A}$, $\ddot{A}$, $\dddot{A}$, &c. will become Fluxions of the firft ordinate A, whilft the Fluxion of the abfcifia z is unity; and then there will come out

$$T = A + \dot{A}z + \tfrac{1}{2}\ddot{A}z^2 + \tfrac{1}{6}\dddot{A}z^3 + \tfrac{1}{24}\ddddot{A}z^4 + \text{\&c.}$$

Therefore the equidiftant ordinates coinciding we happen on a feries where the coefficients of the terms are Fluxions of the firft ordinate refpectively divided by 1, 2, 6, 24, &c. which are generated by a continual multiplication of thefe numbers, 1, 2, 3, 4, &c. And this the learned Dr *Taylor* firft difcover'd in his *Methodus Incrementorum*, and after him, *Hermanus* in his *appendix* to his *Phoronomia*.

Hence if the ordinate of any curve whatfoever be refolved into a feries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where the exponents of the abfcifia z are affirmative whole numbers, the firft term A is the firft ordinate, being that which paffes thro' the beginning of the abfcifia; the two firft $A + Bz$ denote a right line which paffes thro' two coinciding points of the curve; which therefore touches the curve; the three firft terms $A + Bz + Cz^2$ define a conic parabola which paffes thro' three coinciding points of the curve, and for that reafon it touches the curve, and has the fame curvature in that point thro' which the firft ordinate paffes; the four firft terms $A + Bz + Cz^2 + Dz^3$ define a conic parabola which paffes thro' four coinciding points of the curve, that is, which touches the curve, and has the fame curvature and variation of curvature in the point of contact. *Laftly*, the whole feries $A + Bz + Cz^2 + Dz^3 +$ &c. is an ordinate of the parabola of infinite dimenfions, which touches the curve, and in the point of contact has the fame curvature, variation of curvature, variation of variation, and fo *in infinitum*, as *Newton* expreffes it in the 10th propofition of the fecond book of his *Principia*, or, which is all the fame, the whole

B b $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ feries

feries is an ordinate of a parabola paffing thro' equidiftant ordinates of a curve, infinite in number, and coinciding with the firft ordinate.

Hence we have an idea of the analogy which is between the differential method and the common method of feries; the one proceeds by Fluxions, or the ultimate ratios of the differences; and the other generally by the differences of any magnitude whatever.

Let D E F denote any curve, whofe abfciffa A C cuts the equidiftant ordinates A D, B E, C F, at right angles. And let A B=z, A D=A, and from the foregoing B E will be $=A+\dot{A}z+$
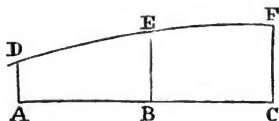
$\frac{1}{2}\ddot{A}z^2+\frac{1}{6}\dddot{A}z^3+\frac{1}{24}\ddddot{A}z^4+$ &c. namely this value of B E is the ordinate of a parabolic curve which coincides with the other curve in the point D: Therefore for the area of the curve, we may ufe the area of the fame parabola, which, by the *Inverfe Method of Fluxions*, comes out A B E D =

$$A z+\tfrac{1}{2}\dot{A}z^2+\tfrac{1}{6}\ddot{A}z^3+\tfrac{1}{24}\dddot{A}z^4+\tfrac{1}{120}\ddddot{A}z^5+ \text{ &c.}$$

and by the fame way, if B E be called *y*, A B being $=$ B C$=z$, the area will be

$$\mathrm{BCFE}=yz+\tfrac{1}{2}\dot{y}z^2+\tfrac{1}{6}\ddot{y}z^3+\tfrac{1}{24}\dddot{y}z^4+\tfrac{1}{120}\ddddot{y}z^5+ \text{ &c.}$$

in which, if the fign of the abfciffa *z* be changed, there will be obtained the area B E D A expreffed negatively, *viz.* by changing the fign of the abfciffa there is obtained the area lying on the other fide of the ordinates. But the area expreffed affirmatively becomes

$$\mathrm{BEDA}=yz-\tfrac{1}{2}\dot{y}z^2+\tfrac{1}{6}\ddot{y}z^3-\tfrac{1}{24}\dddot{y}z^4+\tfrac{1}{120}\ddddot{y}z^5- \text{ &c.}$$
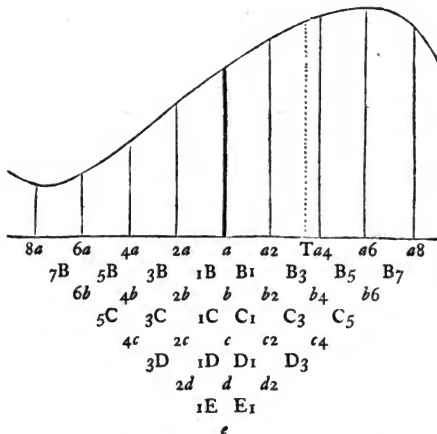
And this is Mr *John Bernoulli*'s feries exhibiting the area expreffed by the laft ordinate and its Fluxions, which we have now given by Fluxions of the firft ordinate. And it is to be obferved, that the firft feries is not extended to thofe cafes in which the firft ordinate touches the curve, nor does *Bernoulli*'s feries extend to thofe cafes wherein the laft ordinate touches the curve. For a parabola, whofe area is ufed for the area of a curve to be fquared, can touch none of the ordinates; fo neither can it coincide with the other curve touching its ordinate. For fuch like expreffions, for areas and ordinates of curves, pre-fuppofe this form of the feries A+Bz+Cz²+ &c. in which the exponents of the abfciffa *z* are affirmative whole numbers.

PROPO-

P R O P O S I T I O N XX.

LET there be given a series of equidistant ordinates running on both ways ad infinitum, *required to find a parabolic line which shall pass thro' the extremities of all of them.*

C A S E I.

LET a denote the ordinate in the midst of all, and let $a2$, $a4$, $a6$, $a8$, &c. be those on one side, and $2a$, $4a$, $6a$, $8a$, &c. be those on the other side, the progression going on each way *in infinitum.* Collect their first differences $7B$, $5B$, $3B$, $1B$, $B3$, $B5$, $B7$; their second $6b$, $4b$, $2b$, b, $b2$, $b4$, $b6$; their third $5C$, $3C$, $1C$, $C1$, $C3$, $C5$; their fourth differences $4c$, $2c$, c, $c2$, $c4$, &c. by taking always the antecedent from the consequent, as in the above proposition.

$8a$		$6a$		$4a$		$2a$		a		$a2$		$Ta4$		$a6$		$a8$
	$7B$		$5B$		$3B$		$1B$		$B1$		$B3$		$B5$		$B7$	
		$6b$		$4b$		$2b$		b		$b2$		$b4$		$b6$			
			$5C$		$3C$		$1C$		$C1$		$C3$		$C5$				
				$4c$		$2c$		c		$c2$		$c4$					
					$3D$		$1D$		$D1$		$D3$						
						$2d$		d		$d2$							
							$1E$		$E1$								
								e									

Now let a, b, c, d, e, &c. be the middle ordinate and differences in alternate orders respectively ; and let $1B$ and $B1$, $1C$ and $C1$, $1D$ and $D1$, $1E$ and $E1$, &c. be the two middle differences in the other orders ; and put $B = 1B + B1$, $C = 1C + C1$, $D = 1D + D1$, $E = 1E + E1$, &c. and let the distance between any ordinate T, and middle ordinate a, be to the common

mon

mon diſtance of the equidiſtant ordinates as z to unity; and the ordinate T will be $=a+$

$$\frac{Bz+bzz}{1.2}+$$

$$\frac{2Cz+czz}{1.2}\times\frac{zz-1}{3.4}+$$

$$\frac{3Dz+dzz}{1.2}\times\frac{zz-1}{3.4}\times\frac{zz-4}{5.6}+$$

$$\frac{4Ez+ezz}{1.2}\times\frac{zz-1}{3.4}\times\frac{zz-4}{5.6}\times\frac{zz-9}{7.8}+$$

$$\frac{5Fz+fzz}{1.2}\times\frac{zz-1}{3.4}\times\frac{zz-4}{5.6}\times\frac{zz-9}{7.8}\times\frac{zz-16}{9.10}+$$

&c.

Where it is to be obſerved that the abſciſſa z is negative, when T the ordinate ſought lies on the contrary ſide of the middle ordinate.

C A S E   II.

LET 1A, and A1, be two middle ordinates; A3, A5, A7, A9, &c. be thoſe on one part; and 3A, 5A, 7A, 9A, &c. thoſe on the other part. Collect their firſt differences 8a, 6a, 4a, 2a, a, a2, a4, a6, a8; the ſecond 7B, 5B, 3B, 1B, B1, B3, B5, B7; their third 6b, 4b, 2b, b, b2, b4, b6, &c. by ſubtracting every where the former from the latter
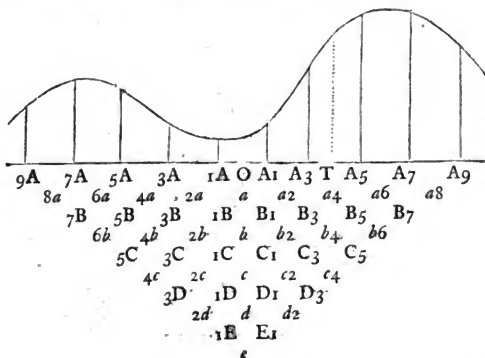
Now

Now take the middle differences a, b, c, d, e, &c. and the two middle differences in the other orders, $viz.$ 1A and A1, 1B and B1, 1C and C1, 1D and D1, 1E and E1, &c. and put A=1A+A1, B=1B+B1, C=1C+C1, D=1D+D1, E=1E+E1, &c. and let the point O be in the middle between the two middle ordinates 1A and A1, and let the distance of every ordinate T from the middle point, namely OT, be to the common interval of the equidistant ordinates as z to 2, and it will be

$$T = \frac{A+az}{2} +$$
$$\frac{3B+bz}{2} \times \frac{zz-1}{4.6} +$$
$$\frac{5C+cz}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} +$$
$$\frac{7D+dz}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} \times \frac{zz-25}{12.14} +$$
$$\frac{9E+ez}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} \times \frac{zz-25}{12.14} \times \frac{zz-49}{16.18} +$$
$$\&c.$$

And in this case also z is affirmative, when T lies on that side of the middle point O, as in the scheme above, and negative when the contrary way. And each case is easily demonstrated, like the foregoing proposition.

E X A M P L E I.

LET there be given five ordinates -3, -8, 1, 12, 37, thro' whose extremities the parabola is to be drawn. Seek their first differences -5, 9, 11, 25; their second 14, 2, 14; their third -12, 12, and the last 24. Then because the number of the ordinates is odd, proceeding to $Case$ I. and by beginning from the middle ordinate, proceed to the middle differences in the alternate orders, by putting $a=1$,

$$
\begin{array}{ccccc}
-3 & -8 & 1 & 12 & 37 \\
 & -5 & 9 & 11 & 25 \\
 & & 14 & 2 & 14 \\
 & & -12 & 12 & \\
 & & & 24 & \\
\end{array}
$$

$b=2$, $c=24$; then B=9+11=20, C=−12+12=0, and these being substituted, there comes out $T=1+\frac{20z+2zz}{1.2}+\frac{24zz}{1.2}\times\frac{zz-1}{3.4}$, or $T=1+10z+z^4$. This is the ordinate of a parabola of four dimensions, passing thro' the extremities of five proposed ordinates, as will appear by writing the numbers -2, -1, 0, 1, 2, successively for z. Here in this case

case

case the curve cuts the base, because the ordinates are part negative and part affirmative.

E X A M P L E II.

Suppose there be given six ordinates, 1, 5, 10, 10, 5, 1 thro' whose extremities it is required to draw the parabola. Seek their differences as in the margin; and because the number of ordinates is even, let the second case be apply'd; then, by beginning from the two middle ordinates, and proceeding to the two middle differences, A will be $=10$ $+10=20$, B$=-5-5=-10$, C$=6$ $+6=12$; then $a=0$, $b=0$, $c=0$; which values being substituted, T

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$4 \quad 5 \quad 0 \quad -5 \quad -4$$
$$1 \quad -5 \quad -5 \quad 1$$
$$-6 \quad 0 \quad 6$$
$$6 \quad 6$$
$$0$$

will be $=\frac{20}{2}-\frac{10}{2}\times\frac{zz-1}{4.6}+\frac{60}{2}\times\frac{zz-1}{4.6}\times\frac{zz-9}{8.10}$, which equation reduc'd

into order becomes T$=\frac{689-50z^2+z^4}{64}$. And to prove the operation,

write -5, -3, -1, 1, 3, 5, successively for z, and there will come out the proposed ordinates. For in the second case of the Proposition, the common interval of the ordinates, or, which is the same, the increment of the abscissa is equal to two.

In this example the powers of the odd dimensions of the abscissa are wanting, because the ordinates from the beginning of the abscissa being from thence equally distant, are of the same sign, and equal amongst themselves. For, here, the equation to the parabola remains the same, tho' the sign of the abscissa be changed. As if the proposed ordinates were $+1$, -5, $+10$, -10, $+5$, -1, or $+1$, $+5$, $+10$, -10, -5, -1, where the ordinates equally distant from the middle are equal, but affected with different signs; in this case, I say, the powers of the abscissa of even dimensions were wanting.

S C H O L I U M.

Concerning the description of a curve of a parabolic kind, thro' any number of given points, several acute geometers since *Newton* have treated; but all their solutions are the same with these exhibited, which differ very little from *Newton*'s, as will be manifest from the fifth lemma of the third Book of his *Principia*, and his *Differential Method* published by the universally learned W. *Jones*, Esq; *Newton* indeed describes a parabola through given points, others have considered the assignation of terms from given differences; but how it may be conceived, or under what form it may

may be exhibited, the problem is the fame. And indeed the invention of the terms, which the values of the ordinate T have, is extremely ingenious, and worthy the excellent author; and after the forms are had, the inveſtigation of the problem is eaſy, in which nothing elſe is required but the reſolution of ſimple equations.

But it is to be obſerved, that the form of the ordinate compoſed of the powers $A + Bz + Cz^2 + Dz^3 +$ &c. which *Newton* aſſumes in demonſtrating the foundation of his method, is badly deſign'd for this purpoſe; for the value of every coefficient comes out into an infinite ſeries: but if any one will aſſume the terms here uſed, he will with very little trouble get the above concluſion.

P R O P O S I T I O N XXI.

G I V E N *a ſeries of primary terms, to find the intermediate ones which are not far diſtant from the beginning.*

Upon a right line given in poſition, at right angles, and at equal diſtances from one another, let there be erected ordinates reſpectively equal to the primary terms; then, by the two foregoing Propoſitions, let there be ſought a parabolic line which paſſes thro' the extremities of them all, and it will alſo paſs thro' the extremities of the intermediate terms; which therefore will be given from the given equation to the parabola. *Q. E. I.*

E X A M P L E I.

SUPPOSE this ſeries $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256},$ &c. to be interpolated, whoſe terms are produced by the continual multiplication of the numbers $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8},$ &c. Seek the differences of the terms, and the differences of the differences, as follows,

$$
\begin{array}{ccccccc}
1 & \frac{1}{2} & \frac{3}{8} & \frac{5}{16} & \frac{35}{128} & \frac{63}{256} & \&c. \\
& -\frac{1}{2} & -\frac{1}{8} & -\frac{1}{16} & -\frac{5}{128} & -\frac{7}{256} & \\
& & \frac{3}{8} & \frac{1}{16} & \frac{3}{128} & \frac{2}{256} & \\
& & & -\frac{5}{16} & -\frac{5}{128} & -\frac{5}{256} & \\
& & & & \frac{35}{128} & \frac{4}{256} & \\
& & & & & -\frac{63}{256} &
\end{array}
$$

For becauſe this ſeries towards one part only runs out *in infinitum*, interpolation muſt be performed by the 19th Propoſition; and if 1 the firſt term be put for the firſt ordinate, A will be $=1$, B$=-\frac{1}{2}$, C$=+\frac{3}{8}$, D$=-\frac{5}{16}$, E$=+\frac{35}{128}$, &c. which being ſubſtituted, there comes out

T$=$

$$T = 1 - \tfrac{1}{4} \times \tfrac{z}{1} + \tfrac{1}{6} \times \tfrac{z}{1} \times \tfrac{z-1}{2} - \tfrac{7}{16} \times \tfrac{z}{1} \times \tfrac{z-1}{2} \times \tfrac{z-2}{3} + \&c.$$

that is, $T = 1 - \tfrac{1}{4} A \tfrac{z}{1} - \tfrac{1}{6} B \tfrac{z-1}{2} - \tfrac{7}{6} C \tfrac{z-2}{3} - \tfrac{7}{4} D \tfrac{z-3}{4} - \&c.$

where A, B, C, D, &c. now denote the terms of this series, according to *Newton*'s method; but every primary term may be taken for the first ordinate, suppose for instance the second be taken, and it will be

$$A = \tfrac{1}{4},\ B = -\tfrac{1}{4},\ C = +\tfrac{1}{16},\ D = -\tfrac{7}{16},\ E = +\tfrac{7}{16},\ \&c.\ \text{Therefore}$$

$$T = \tfrac{1}{4} - \tfrac{1}{4} \times \tfrac{z}{1} + \tfrac{1}{16} \times \tfrac{z}{1} \times \tfrac{z-1}{2} - \&c.$$

that is, $T = \tfrac{1}{4} - \tfrac{1}{4} A \tfrac{z}{1} - \tfrac{1}{6} B \tfrac{z-1}{2} - \tfrac{7}{6} C \tfrac{z-2}{3} - \tfrac{7}{16} D \tfrac{z-3}{4} - \&c.$

But we must know that z is the distance between the term sought, and that which is used for the first ordinate. As if T the desired term stand in the middle between the first and second, put $+\tfrac{1}{2}$ for z in the first value of T, and $-\tfrac{1}{2}$ for z in the last value, and for that term you will have the two following series,

$$1 - \tfrac{1}{8} A + \tfrac{3}{16} B + \tfrac{15}{16} C + \tfrac{11}{16} D + \tfrac{63}{16} E + \&c.$$

$$\tfrac{1}{4} + \tfrac{1}{8} A + \tfrac{5}{16} B + \tfrac{21}{16} C + \tfrac{45}{16} D + \tfrac{7}{16} E + \&c.$$

And because these series converge very slowly, they must be summed by the *Theorem* in *Scholium* of the eleventh Proposition; where we must know the values of the terms come out the most simple, when the term which stands next to the intermediate one sought, is put for the first ordinate. But when the term sought is very far distant from the beginning, the 18th Proposition must be used, as shall be shown in the following examples.

E X A M P L E II.

LET this series 1, 1, 2, 6, 24, 120, 720, &c. be interpolated, whose terms are generated by the continual multiplication of the numbers 1, 2, 3, 4, 5, 6, &c. Forasmuch as these terms increase very fast, their differences will make a diverging progression, which hinders the ordinate of the parabola from approaching to the truth; therefore in this and the like cases, I interpolate the logarithms of the terms, whose differences constitute a series swiftly converging, though the terms increase very fast, as in the present example.

I now propose the finding of a term which stands in the midst between the two first, 1 and 1 ; and because the logarithms of the initial terms have

<div align="right">differences</div>

differences flowly converging, I firft feek the term ftanding in the middle, between two fufficiently remote from the beginning, for inftance, between the eleventh 3628800 and the twelfth 3991680o; and from that being given, I can go back to the term fought, by the 16th Propofition. And when any terms can be had, ftanding on each fide of the intermediate one, which muft firft be found, I purfue the operation by the fecond cafe of the twentieth Propofition. For when the computation is not in fpecies, but in numbers, I can proceed by this method, as often as the numbers of terms ftanding on each fide of the term fought is given fufficiently great, though the feries to be interpolated may not indeed run out on each fide *in infinitum*.

I now take out from a table the logarithms of twelve terms, whofe firft is that of the fixth term 120; fo that the fix before, and as many after that which is required, may be had; than becaufe the defired term ftands directly in the middle of all, the abfciffa z will be =o, by the fecond cafe of the twenty firft Propofition; and therefore the firft, fecond, third, &c. differences of the odd orders which are multiplied into z, will not enter into the computation; therefore I gather only the fecond, fourth, &c. differences in even orders, as you fee.

Logarithms	2d Differen.	4th Differ.	6th Diff.	8th Diff.	10th Diff.
2.0791812460					
2.8573324964	669467896				
3.7024305364	579991470	21154180			
4.6055205234	511525224	14443928	2568588		
5.5597630329	457574906	10302264	1446210	541511	
6.5597630329	413926852	7606810	865343	259252	156590
7.6011557180	377885608	5776699	543728	133583	65082
8.6803369641	347621663	4490316	355696	72996	
9.7942803164	321846834	3559629	240660		
10.9404083521	299632234	2869602			
12.1164996111	280287236				
13.3206195938					

Now I take out the two middle logarithms, and the two middle differences and their fums I put equal to A, B, C, D, &c. refpectively, as you fee

$$
\begin{array}{ccc}
6.5597630329 & 413926852 & 7606810 \\
7.6011557180 & 377885608 & 5776699 \\
A=14.1609187509; & B=791812460, & C=13383509 \\
\end{array}
$$

D d

865343

$$
\begin{array}{ccc}
865343 & 259252 & 156590 \\
543728 & 133583 & 65082 \\
\hline
D=1409071, & E=392835, & F=221072,
\end{array}
$$

Then in the second cafe of the twenty firft Propofition, I fubftitute 0 for z, and I have

$$T = \tfrac{1}{2}A - \tfrac{1}{8}B + \tfrac{1}{16}C - \tfrac{5}{2048}D + \tfrac{11}{65536}E - \tfrac{61}{524288}F + \&c.$$

Write now the values of A, B, C, D, E, F juft now found, and the computation will be as follows,

$$
\begin{array}{cc}
7.08045937545 & 494882787 \\
1568380 & 34401 \\
2098 & 266 \\
\hline
+7.08047508023 & -494917454
\end{array}
$$

And by fubducting the fum of the negative terms from that of the affirmative, there will remain $T = 7.07552590569$; and this is the Logarithm of the number 11899423.08, being that which ftands in the middle between the terms 3628800 and 39916800.

Now as the primary terms are formed by drawing the firft continually into the numbers 1, 2, 3, 4, &c. fo, by the fixteenth Propofition, the intermediate terms are generated by drawing the intermediate term between the firft and fecond into the numbers $1\tfrac{1}{2}$, $2\tfrac{1}{2}$, $3\tfrac{1}{2}$, $4\tfrac{1}{2}$, &c. continually. For inftance, the product under the ten factors $1\tfrac{1}{2}$, $2\tfrac{1}{2}$, $3\tfrac{1}{2}$, $4\tfrac{1}{2}$, $5\tfrac{1}{2}$, $6\tfrac{1}{2}$, $7\tfrac{1}{2}$, $8\tfrac{1}{2}$, $9\tfrac{1}{2}$, $10\tfrac{1}{2}$, and the term which ftands in the middle between the firft and fecond, is equal to the intermediate term 11899423.08 juft now found, whofe place is in the middle between the eleventh and twelfth. Therefore if the intermediate term be divided by $10\tfrac{1}{2}$, and that quotient by $9\tfrac{1}{2}$, and the laft quotient by $8\tfrac{1}{2}$, and fo on to the divifor $1\tfrac{1}{2}$, the laft of the quotients will be equal to a term in the midft between 1 and 1. But the intermediate terms coming out by that divifion are

Primaries

Primaries	Intermediates
39916800	
	—11899423.08
3628800	
	—1133278.389
362880	
	—119292.4620
40320	
	—14034.40729
5040	
	—1871.254305
720	
	—287.88552777
120	
	—52.34277777
24	
	—11.63172839
6	
	—3.323350969
2	
	—1.329340388
1	
	—.8862269251
1	
	—1.7724538502

Whence it appears that the term between 1 and 1 is .8862269251, whose square is .7853 ... &c. namely the area of a circle whose diameter is unity ; and the double of it is 1.7724538502, *viz.* that standing before the first primary at half the common distance, is equal to the square root of 3.1415926 ... &c. which denotes the circumference of a circle, whose diameter is unity. For if the squares of the primary numbers constitute a new series 1, 1, 4, 36, 576, 14400, &c. the term in the middle between the first and second will be to unity, as the area of a circle to its circumscrib'd square : And the term which stands before the first at half of the common distance will be to unity as the circumference of a circle to the diameter. But in the following pages we will show that such like series may be interpolated without logarithms.

E X A M P L E   III.

LET a curve be squared whose ordinate is $x^{\theta-1} \times \overline{e + f x^{\eta}}|^{\lambda}$. Write 0, 2, 3, 4, &c. successively for the index λ, and their will come out a series of equidistant ordinates $x^{\theta-1} \times \overline{e + f x^{\eta}}|^{0}$, $x^{\theta-1} \times \overline{e + f x^{\eta}}|^{1}$, $x^{\theta-1} \times \overline{e + f x^{\eta}}|^{2}$, &c. among which, the proposed ordinate obtains a place distant from the beginning by the distance λ. Therefore the area sought will likewise obtain the same place between the areas of those ordinates which constitute the following progression of equidistant terms,

$\frac{1}{\theta} x$

$$\frac{1}{\theta}x^\theta, \quad \frac{e}{\theta}x^\theta + \frac{f}{\theta+n}x^{\theta+n}, \quad \frac{e^2}{\theta}x^\theta + \frac{2ef}{\theta+n}x^{\theta+n} + \frac{f^2}{\theta+2n}x^{\theta+2n}, \&c.$$

Now if thefe areas be interpolated by the nineteenth propofition, there would come out for the area fought, the fame feries as by the common method for reducing an ordinate into a converging feries, that the Fluent from thence may be found. Neverthelefs if the terms in the feries of areas be firft divided by the terms of this geometrical progreffion refpectively, namely $\overline{e+fx^n}\vert^0$, $\overline{e+fx^n}\vert^1$, $\overline{e+fx^n}\vert^2$, $\overline{e+fx^n}\vert^3$, &c. that is, if there be put

$$A = x^\theta \times \frac{\frac{1}{\theta}}{\overline{e+fx^n}\vert^0}$$

$$A_1 = x^\theta \times \frac{\frac{e}{\theta} + \frac{fx^n}{\theta+n}}{\overline{e+fx^n}\vert^1}$$

$$A_2 = x^\theta \times \frac{\frac{e^2}{\theta} + \frac{2efx^n}{\theta+n} + \frac{f^2x^{2n}}{\theta+2n}}{\overline{e+fx^n}\vert^2},$$

$$A_3 = x^\theta \times \frac{\frac{e^3}{\theta} + \frac{3e^2fx^n}{\theta+n} + \frac{3ef^2x^{2n}}{\theta+2n} + \frac{f^3x^{3n}}{\theta+3n}}{\overline{e+fx^n}\vert^3},$$
$$\&c.$$

and if a computation be purfu'd according to the nineteenth Propofition, the differences will be found

$$B = \frac{-nfx^{\theta+n}}{\theta.\theta+n.\overline{e+fx^n}}, C = \frac{+2n^2f^2x^{\theta+2n}}{\theta.\theta+n.\theta+2n.\overline{e+fx^n}\vert^2}, D = \frac{-6n^3f^3x^{\theta+3n}}{\theta.\theta+n.\theta+2n.\theta+3n.\overline{e+fx^n}\vert^3}$$

&c, which being fubftituted for A, B, C, D, &c. and λ for z, you will find the term remote from the beginning by the diftance λ, to be

$$\frac{x^\theta}{\theta} - \frac{\lambda nfx^{\theta+n}}{\theta.\theta+n.\overline{e+fx^n}} + \frac{\lambda.\overline{\lambda-1}.n^2f^2x^{\theta+2n}}{\theta.\theta+n.\theta+2n.\overline{e+fx^n}\vert^2} - \frac{\lambda.\overline{\lambda-1}.\overline{\lambda-2}.n^3f^3x^{\theta+3n}}{\theta.\theta+n.\theta+2n.\theta+3n.\overline{e+fx^n}\vert^3} +$$

&c. But becaufe the terms to be interpolated were divided by the powers of $e+fx^n$, every one, to wit, by that power, whofe index was the diftance of that term from the beginning; on the contrary draw the term now found into the power of the aforefaid binomial, whofe index is λ, as being

being its distance from the beginning, and we shall have for the area of the curve

$$\frac{x^\theta}{\theta} \times \overline{e+fx^n}|^\lambda \text{ into } 1 - \frac{\lambda n f x^n}{\theta+n \cdot \overline{e+fx^n}} + \frac{\lambda \cdot \overline{\lambda-1} \cdot n^2 f^2 x^{2n}}{\theta+n \cdot \theta+2n \cdot \overline{e+fx^n}|^2} \&c.$$

or by putting $y = \frac{fx^n}{e+fx^n}$, $r = \frac{\theta+n}{n}$, and by writing the series according to *Newton*'s way, there will come out

$$\frac{x^\theta}{\theta} \times \overline{e+fx^n}|^\lambda - \frac{\lambda}{r}Ay - \frac{\lambda-1}{r+1}By - \frac{\lambda-2}{r+2}Cy - \frac{\lambda-3}{r+3}Dy - \&c.$$

for the area of a curve whose ordinate in general is $x^{\theta-1} \times \overline{e+fx^n}|^\lambda$. And this series, transformed by the seventh proposition, will become that of *Newton*'s for the quadrature of a binomial curve. The series terminates when the index λ is an affirmative whole number; and, after a due preparation of the ordinate it will always terminate when the curve is quadrable; but its chief use is, that it may exhibit the areas in a series very simple. If the coefficients e, f be affected with contrary signs, *Newton*'s series is preferable; but ours when they are like signs.

E X A M P L E   IV.

REQUIRED to assign the uncias of a binomial from a given medial *uncia* in that power whose index is an even number.

If n denote the index of the power, and the medial uncia be drawn continually into the fractions $\frac{n}{n+2}$, $\frac{n-2}{n+4}$, $\frac{n-4}{n+6}$, &c. the products will be the remaining uncias standing on each side of the medial one.

$$a6 = a \times \frac{n}{n+2} \times \frac{n+2}{n+4} \times \frac{n+4}{n+6},$$

$$a4 = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$a2 = a \times \frac{n}{n+2},$$

$$a = a,$$

$$2a = a \times \frac{n}{n+2},$$

$$4a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$6a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4} \times \frac{n-4}{n+6}.$$

E e

And

And if a denote the medial uncia, then $a2$, $a4$, $a6$, &c. on one side; and $2a$, $4a$, $6a$, &c. on the other, will denote the other uncias. Then by making a computation by the first case of the twentieth proposition, you will find

$$a - \frac{rr}{2.n+2}A - \frac{rr-4}{4.n+4}B - \frac{rr-16}{6.n+6}C - \frac{rr-36}{8.n+8}D - \&c.$$

is the term of the series to be interpolated, whose distance from the middle term a is to the common distance of the primary ones as r to 2. Suppose, for example, in the twelfth power, the *uncias* are 1, 12, 66, 220, 495, 792, 924, 792, &c. the middle uncia a being $= 924$. And if that be required which is 3 distant from the medial uncia, r will be $= 6$; which being substituted, and 12 for n, there comes out

$$924 - \frac{36}{2.14}A - \frac{32}{4.16}B - \frac{20}{6.18}C,$$

for the uncia required, the series terminating; and these terms freed from fractions are $924 - 1188 + 594 - 110$, whose sum under their proper signs is 220, which is the value of the uncia sought.

And after the same manner if the index n be an odd number, and A another of the medial uncias; then will

$$A - \frac{rr-1}{2.n+3}A - \frac{rr-9}{4.n+5}B - \frac{rr-25}{6.n+7}C - \frac{rr-49}{8.n+9}D - \&c.$$

be the uncia whose distance from the intermediate point between the two middle uncias, is to the common distance, as r to 2.

And these series, in a very great power, will converge when the distance between the middle uncia and that sought is but small in respect of the index of the power.

S C H O L I U M.

AFTER a series to be interpolated is duly prepared by the seventeenth Proposition, though it may on each side run out *in infinitum*, we may proceed by the nineteenth Proposition, except when the terms equidistant from the middle one, are equal among themselves: When this happens, we must apply the first Case of the twentieth Proposition, if any primary term by any law claim a place in the middle of all; or if two terms by the same law claim the middle place, the second Case of the same Proposition must be used. And in other cases we may almost proceed at pleasure.

P R O-

P R O P O S I T I O N XXII.

G*IVEN a series of equidistant terms, to find any primary or interme-diate term how far foever distant from the beginning of the series.*

If the term fought be far diftant from the beginning, then by the eighteenth Propofition feek another feries, wherein the defired term fhall make a term near the beginning ; then proceed as in the above Propofition.

E X A M P L E I.

Let it be propofed to find any term of this feries $1, \frac{1}{1}A, \frac{1}{2}B, \frac{1}{3}C, \frac{1}{4}D$, &c. diftant from the beginning by any fpace m, however great. By the eighteenth Propofition, the term of the feries

$$1, \quad \frac{a}{m+1}, \quad \frac{2b}{m+2}, \quad \frac{3c}{m+3}, \quad \frac{4d}{m+4}, \quad \&c.$$

which ftands before the firft, by half of the common diftance, will be equal to that term of the firft feries, whofe diftance from the beginning is m. But it appears by the fecond example of the twenty firft Propofition, that the term diftant by half of the common diftance before the firft in the feries of the numerators $1, 1.1, 1.1.2, 1.1.2.3$, &c. that is, in this this feries $1, 1, 2, 6, 24, 120$, &c. is the fquare root of the number $3.1415926 \ldots$ &c. wherefore I only interpolate the denominators, namely

$$1, \quad \frac{1}{m+1}, \quad \frac{1}{m+1.m+2}, \quad \frac{1}{m+1.m+2.m+3}, \quad \&c.$$

And becaufe this feries may be continued on both ways *in infinitum*, let it actually be continued ; and it will be

$$\&c. \; \overline{m-2 . m-1 . m}, \quad \overline{m-1 . m}, \; m, \; 1, \; \frac{1}{m+1}, \quad \frac{1}{m+1.m+2}, \quad \&c.$$

where the term required ftands in the middle between the two midle ones m and 1, but becaufe the differences of thefe terms are very great, let thofe that are equidiftant from the middle be drawn into one another, that is, m into 1 ; $\overline{m-1.m}$ into $\frac{1}{m+1}$, and fo on ; and there will come out a new feries

$$\&c. \; \frac{m-1}{m+2} . \frac{m-1}{m+1} . m, \; \frac{m-1}{m+1} m, \; m, \; m, \; m \frac{m-1}{m+1}, \; m \frac{m-1}{m+1} . \frac{m-2}{m+2}, \quad \&c.$$

running out on each fide *in infinitum*, and having thofe terms equal which

are

are equally diſtant from the middle.   But the term in the middle, ſtanding between the two middle primary ones m and m, is the ſquare of that in the middle between m and 1 in the former ſeries.   In the laſt ſeries therefore ſeek the term between m and m by the ſecond caſe of the twentieth Propoſition, and you will find it to be

$$m + \frac{m}{4 \cdot m + 1} + \frac{9m}{32 \cdot m + 1 \cdot m + 2} + \frac{75m}{128 \cdot m + 1 \cdot m + 2 \cdot m + 3} + \&c.$$

which draw into the circumference of a circle, namely into the reſpective term in the ſeries of numerators, and you will have for the ſquare of the term ſought $3.14159 \dots \&c.$ into

$$m + \frac{A}{4 \cdot m + 1} + \frac{9B}{8 \cdot m + 2} + \frac{25C}{12 \cdot m + 3} + \frac{49D}{16 \cdot m + 4} + \&c.$$

Therefore the term of the propoſed ſeries $1, \frac{2}{3}A, \frac{4}{5}B, \frac{6}{7}C, \&c.$ diſtant by the interval m from the beginning, is equal to a mean proportional between the circumference of a circle, and that ſeries; which converges by ſo much the ſwifter, as m is greater, that is, by how much the term ſought is further diſtant from the beginning.

Example.   Suppoſe $n = 100$, and the firſt term of the ſeries will be the circumference of a circle drawn into 100, or $A = 314.15927$;

then will $B = \dfrac{A}{4.101} = .77762$, $C = \dfrac{9B}{8.102} = .00858$, $D = $

$\dfrac{25C}{12.103} = .00017$, and the ſum of theſe four terms is 314.94564, whoſe ſquare root 17.7467079 is the hundred and firſt term of the ſeries to be interpolated; or the product under the factors $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9}$, $\&c.$ whoſe number is a hundred. And in the ſame manner we may find any intermediate term; for if $99\frac{1}{2}$ be wrote for m, we ſhall have the term in the middle between the hundredth and hundredth and firſt; or if $99\frac{1}{3}$ be ſubſtituted for m, we ſhall have the term ſtanding after the hundredth by a third part of the common diſtance.

$$\begin{aligned}&314.15927\\&77762\\&858\\&17\\ \hline &314.94564\end{aligned}$$

Whence it is to be noted, that the reciprocals of the terms of any ſeries may be interpolated: Thus the reciprocals of the terms in the laſt ſeries conſtitute the ſeries $1, \frac{1}{2}A, \frac{1}{3}B, \frac{1}{4}C, \frac{1}{5}D, \&c.$ wherein the term remote from the beginning by any equal diſtance $\frac{1}{2}m$, will be equal to the term of the ſeries

$$1, \frac{a}{m + 1}, \frac{3b}{m + 3}, \frac{5c}{m + 5}, \frac{7d}{m + 7}, \&c.$$

which ſtands in the middle between the firſt and ſecond term; and which

will therefore be a mean proportional between the following series

$$\frac{1}{m+1} + \frac{A}{2.m+3} + \frac{9B}{4.m+5} + \frac{25C}{6.m+7} + \frac{49D}{8.m+9} + \&c.$$

and the number .6366197723676, which is equal to unity divided by the semicircumference of a circle: This will be manifest by pursuing the steps of the first part of this example.

E X A M P L E   II.

REQUIRED the term of this series 1, $\frac{2}{4}A$, $\frac{4}{7}B$, $\frac{6}{7}C$, $\frac{14}{16}D$, &c. how far soever remote from the beginning, namely by the distance m; by the eighteenth Proposition that term will be equal to the term of this series,

$$1, \quad \frac{2a}{3m+2}, \quad \frac{5b}{3m+5}, \quad \frac{8c}{3m+8}, \quad \frac{11d}{3m+11}, \quad \&c,$$

which stands before the first by a third part of the common distance; therefore the numerators and denominators may be intercalated separately, as in the second example of the twenty first Proposition, namely by logarithms, and we shall have the term sought.

S C H O L I U M.

HENCE it appears that the terms of series, tho' never so far distant, may be found as accurately as the intermediate terms near the beginning; but in the series 1, $\frac{r}{p}A$, $\frac{r+1}{p+1}B$, &c. to be interpolated, if the difference between p and r be great, it will be troublesome to find any term; but the case is very easy when $p-r$ is $=\pm\frac{1}{2}$, as in the first example, except when that difference is a whole number, in which case the series will be accurately interpolable.

P R O P O S I T I O N   XXIII.

TO find the ratio which the middle uncia bears to the sum of all the uncias in any power of the binomial.

F I R S T   S O L U T I O N.

IF the index of the power be an even number, call it n; if it be odd, call it $n-1$; and it will be as the middle uncia to the sum of them all of the same power, so is unity to a mean proportional between the semicircumference of a circle, and either of the following series,

$n+$

$$n + \frac{A}{2.n+2} + \frac{9B}{4.n+4} + \frac{25C}{6.n+6} + \frac{49D}{8.n+8} + \frac{81E}{10.n+10} + \&c.$$

$$\text{or } n+1 - \frac{A}{2.n-1} - \frac{9B}{4.n-3} - \frac{25C}{6.n-5} - \frac{49D}{8.n-7} - \frac{81E}{10.n-9} \&c.$$

For instance, suppose the ratio of the middle uncia to the sum of all in the hundredth or in the ninety ninth power, be required, n will be $=100$; which multiplied into the semiperiphery of a circle produces the first term A$=$ 157.079632679; then will be B$=\frac{A}{204}$, C$=\frac{9B}{416}$, D$=$ $\frac{25C}{636}$ &c. and by compleating the computation as in the margin, you will find the sum of the terms to be 157.866984459, whose square root 12.5645129018 is to unity as the sum of all the uncias to the middle one in the hundredth or ninety ninth power. And this is the computation by the first series: for though there be little difference, I prefer that wherein the terms are all of the same sign.

157.079632679
769998199
16658615
654820
37137
2734
246
26
3
157.866984459

The *latter* S O L U T I O N.

LET n remain as before; then the sum of all the uncias will be to the middle uncia in the subduplicate ratio of the semiperiphery of a circle to either of the following series,

$$\frac{1}{n+1} + \frac{A}{2.n+3} + \frac{9B}{4.n+5} + \frac{25C}{6.n+7} + \frac{49D}{8.n+9} + \frac{81E}{10.n+11} + \&c.$$

$$\text{or } \frac{1}{n} - \frac{A}{2.n-2} - \frac{9B}{4.n-4} - \frac{25C}{6.n-6} - \frac{49D}{8.n-8} - \frac{81E}{10.n-10} + \&c.$$

or, which comes to the same, put $a = .6366197723676$, namely the quotient which is made by dividing unity by the semicircumference of a circle; and a mean proportional between a and either of the above series will be to unity, as the middle uncia to the sum of them all.

Suppose the index $n=100$, as in the above computation it will be, by the first series, A$=\frac{a}{101}$, B$=\frac{A}{206}$, C$=\frac{9B}{420}$, D$=\frac{25C}{642}$, &c. and from the annex'd calculation it appears that the sum of the terms is .0063344670787,

.0063030166c630 5
3059789351
6556914
2553829
143273
10469
934
98
12
2
.0063344670787.

whose

whose square root .0795892373872 is to unity as the middle uncia to the sum of them all in the ninety ninth or the hundredth power.

And so all the four series are equally simple for the solution of this problem; but in practice there is no need to recur to series; for it is sufficient to take a mean proportional between the semicircumference of a circle and $n+\frac{1}{2}$; for this will always approximate nearer than the two first terms of the series, whose first alone is for the most part sufficient; for instance, suppose $n=100$, then will $n+\frac{1}{2}=100\frac{1}{2}$, which drawn into the femicircumference of a circle produces 157.865, whose square root is 12.5644, deficient by unity in the last figure.

But the same approximation may be made otherwise, and perhaps more commodious for practice. Put c to unity, as the square of the diameter is to the circle; that is, let $c=1.2732395447352$, and the sum of the uncias to the middle one will be as unity to $\sqrt{\dfrac{c}{2n+1}}$ very nearly; the error being too much by about $\dfrac{1}{16nn}\sqrt{\dfrac{c}{2n+1}}$. Let $n=100$, then will $\dfrac{c}{2n+1}=.006334525$, and its square root .07958973 is accurate in the sixth decimal: but if it be divided by $16nn$, that is, by 160000, will give the correction .00000050; and this subducted from the approximation, leaves .07958923 the number sought accurate in the last figure.

Likewise let $n=900$, then $\dfrac{c}{2n+1}=.000706962545$, whose square root .026588767 is too much by 2 in the last figure. But if a correction be computed and subtracted from the approximation, we shall have the number sought accurate in the thirteenth decimal.

See here an approximation equally easy and more accurate. Let the difference between the logarithms of the numbers $n+2$ and $n-2$, be divided by 16, and add the quotient to half the logarithm of the index n, then to this sum add the constant logarithm .0980599385151, namely half of the logarithm of the semicircumference of a circle, and the last sum will be the logarithm of the number which is unity as the sum of the uncias to the middle one. *Example*, suppose $n=900$, the computation will be

$\frac{1}{2}l$, 900	1.4771212547
16) diff. log. 902 and 898	.0001206376
Log. constant	.0980599385
Sum	1.5753018308

And this sum is too much by 2, in the last figure, and is the logarithm of the

the number 37.6098698, which is to unity as the sum of the uncias to the middle one in the power 900 or 899. And if you will have the reciprocal of the number, take the complement of the logarithm, namely — 2.4246981692, and the number corresponding to it, will be .0265887652, which shews the ratio of the middle uncia to the sum of them all in the above said powers.

DEMONSTRATION.

THE powers of a binomial, whose indices are even numbers, have one middle uncia; but those whose indices are odd, have two middle uncias. And hence arise two cases of the problem: First, when the index is even, divide the sums of the uncias 1, 4, 16, 64, 256, 1024, &c. by their middle uncias 1, 2, 6, 20, 70, 252, &c. and the quotients 1, 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, $\frac{256}{63}$, &c. or 1, $\frac{1}{1}$A, $\frac{1}{1}$B, $\frac{6}{1}$C, $\frac{1}{1}$D, &c. will be to unity as the sums of the uncias to the middle uncias in these different powers.

Likewise if the sums of the uncias in odd powers, namely 2, 8, 32, 128, 512, &c. be divided by their middle uncias 1, 3, 10, 35, 126, &c. the same quotes will be produced again, namely 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, &c. For there is the same relation between the sum of the uncias and the middle uncia in any even power, as there is between the sum of the uncias, and the middle uncia in the odd power next below it. Consequently the interpolation of the series 1, $\frac{1}{1}$A, $\frac{1}{1}$B, $\frac{6}{1}$C, $\frac{1}{1}$D, &c. as in the first example of the twenty second Proposition, solves either case of the problem. But we will here give the investigation of these series without the differential method.

The Analysis of the first SOLUTION.

THE series 1, $\frac{1}{1}$A, $\frac{1}{1}$B, &c. to be interpolated, is defined by the equation $T' = \frac{n+2}{n+1} T$, where n is a variable quantity, and its successive values 0, 2, 4, 6, 8, &c. namely, the indices of the powers when even, or the indices increased by unity, when odd. Square each part of the equation to be resolved, and you will have $T'T' = \frac{nn+4n+4}{nn+2n+1} TT$, or, which is the same, $2T'T' + \overline{n+2} \times \overline{TT - T'T'} - \frac{T'T'}{n+2} = 0$. Assume now

$$TT = An + \frac{Bn}{n+2} + \frac{Cn}{n+2 \cdot n+4} + \frac{Dn}{n+2 \cdot n+4 \cdot n+6} + \&c.$$

and after a due reduction, according to the precepts already laid down, you will find

$$TT =$$

$$TT = An + B + \frac{C - 2B}{n+2} + \frac{D - 4C}{n+2 \cdot n+4} + \&c.$$

In which write the confequent values of the variable quantities for the antecedents, that is, $T'T'$ for TT, and $n+2$ for n, and there will come out

$$T'T' = A\overline{n+2} + B + \frac{C - 2B}{n+4} + \frac{D - 4C}{n+4 \cdot n+6} + \&c.$$

Then by taking the difference of thefe values, and drawing it into $n+2$, there will come out

$$\overline{n+2} \times \overline{TT - T'T'} = -2A\overline{n+2} * + \frac{2C - 4B}{n+4} + \frac{4D - 16C}{n+4 \cdot n+6} + \&c.$$

But in the value of TT before affumed, if $n+2$ be wrote for n, we fhall have

$$T'T' = \overline{n+2} \text{ into } A + \frac{B}{n+4} + \frac{C}{n+4 \cdot n+6} + \frac{D}{n+4 \cdot n+6 \cdot n+8} + \&c.$$

And by dividing by $n+2$,

$$\frac{T'T'}{n+2} = A + \frac{B}{n+4} + \frac{C}{n+4 \cdot n+6} + \frac{D}{n+4 \cdot n+6 \cdot n+8} + \&c.$$

Subftitute now in the equation to be refolved, the values of $T'T'$, $\overline{n+2} \times \overline{TT - T'T'}$ and $\frac{T'T'}{n+2}$ reduced to the fame form, and there will refult

$$2B - A + \frac{4C - 9B}{n+4} + \frac{6D - 25C}{n+4 \cdot n+6} + \frac{8E - 49D}{n+4 \cdot n+6 \cdot n+8} + \&c. = 0.$$

Laftly by putting the numerators equal to nothing, you will have $2B - A = 0$, $4C - 9B = 0$, $6D - 25C = 0$, $8E - 49D = 0$, &c. And thefe are the relations of the coefficients in the firft feries; and the latter feries in the former folution is found after the fame manner.

The Analyfis of the latter SOLUTION.

THE latter folution is performed by interpolating the feries, $1, \frac{1}{2}a, \frac{1}{4}b, \frac{1}{6}c, \frac{1}{8}d$, &c. whofe terms are the reciprocals of thofe in the firft; and it is defined by the equation $T' = \frac{n+1}{n+2}T$, in which the fucceffive values of n are $0, 2, 4, 6, 8$, &c. as before; and by fquaring it, it becomes $T'T' = \frac{nn + 2n + 1}{nn + 4n + 4}TT$, that is, $\overline{n+1 \cdot n+3} \; \overline{TT - T'T'} - 2 \cdot \overline{n+1}TT - T'T' = 0$.

Suppofe now

G g $TT =$

$$TT = \frac{A}{n+1} + \frac{B}{n+1.\,n+3} + \frac{C}{n+1.\,n+3.\,n+5} + \frac{D}{n+1.\,n+3.\,n+5.\,n+7} +$$
&c.

And by substituting $n+2$ for n, there will come out

$$T'T' = \frac{A}{n+3} + \frac{B}{n+3.\,n+5} + \frac{C}{n+3.\,n+5.\,n+7} + \frac{D}{n+3.\,n+5.\,n+7.\,n+9} +$$
&c.

Therefore

$$\overline{n+1.\,n+3}.\overline{TT - T'T'} = 2A + \frac{4B}{n+5} + \frac{6C}{n+5.\,n+7} + \frac{8D}{n+5.\,n+7.\,n+9} +$$
&c.

which reduced to a due form, becomes

$$2A + \frac{4B}{n+3} + \frac{6C-8B}{n+3.\,n+5} + \frac{8D-24C}{n+3.\,n+5.\,n+7} + \&c.$$

Write now these values in the equation to be solv'd, and there will result

$$\frac{2B-A}{n+3} + \frac{4C-9B}{n+3.\,n+5} + \frac{6E-25C}{n+3.\,n+5.\,n+7} + \frac{8E-49D}{n+3.\,n+5.\,n+7.\,n+9} + \&c. = 0.$$

Where the numerators being put equal to nothing, will give the relation to the coefficients of the former series in the latter solution.

But that the coefficient A is in one case the semi-circumference of a circle, and reciprocal of it in the other, is demonstrated thus. By the first series $TT = An$ into $1 + \frac{1}{2.\,n+2} + \&c.$ and the greater n is, the nearer will the equation $TT = An$ approach to the truth, because the latter terms will at last be infinitely less than the former. Therefore in the equation $TT = An$, or $A = \frac{TT}{n}$, if for n successively be wrote its values 2, 4, 6, 8, 10, &c. and also for TT the squares of the correspondent terms, there will arise equations continually approximating to the true value of A, namely,

$$A = 2,$$
$$A = 2 \times \tfrac{8}{9},$$
$$A = 2 \times \tfrac{8}{9} \times \tfrac{24}{25},$$
$$A = 2 \times \tfrac{8}{9} \times \tfrac{24}{25} \times \tfrac{48}{49},$$
&c.

Wherefore the value of A is the product under all these $2 \times \tfrac{8}{9} \times \tfrac{24}{25} \times \tfrac{48}{49} \times \tfrac{80}{81} \times$ &c. *in infinitum*, which is equal to the circumference of a circle by Dr *Wallis*'s *Arithmetic of Infinites*.

P R O-

PROPOSITION XXIV.

IF in the ordinate of a curve $x^{r+z-1} \times \overline{1-x}^{p-r-1}$ be wrote successively the whole nnmbers 0, 1, 2, 3, 4, &c. for z; I say, that there is the same relation between the areas of the ordinates coming out, as there is between the terms of the series a, $\frac{r}{p}a$, $\frac{r+1}{p+1}b$, $\frac{r+2}{p+2}c$, $\frac{r+3}{p+3}d$, &c. when the abscissa x is equal to unity.

For let the areas and correspondent ordinates be

Areas.	Ordinates.
A	$x^{r-1} \times \overline{1-x}^{p-r-1}$
B	$x^r \times \overline{1-x}^{p-r-1}$
C	$x^{r+1} \times \overline{1-x}^{p-r-1}$
D	$x^{r+2} \times \overline{1-x}^{p-r-1}$
E	$x^{r+3} \times \overline{1-x}^{p-r-1}$
&c.	&c.

Then, by comparing these areas by the seventh proposition of *Newton*'s quadrature of Curves, you will find

$$B = \frac{rA - x^r \times \overline{1-x}^{p-r}}{p},$$

$$C = \frac{\overline{r+1}B - x^{r+1} \times \overline{1-x}^{p-r}}{p+1},$$

$$D = \frac{\overline{r+2}C - x^{r+2} \times \overline{1-x}^{p-r}}{p+r},$$

$$E = \frac{\overline{r+3}D - x^{r+3} \times \overline{1-x}^{p-r}}{p+3},$$

&c.

Let x now be equal to unity, as it is put in the theorem; then $1-x$ will be $=0$, in which case the relation of the areas becomes $B = \frac{r}{p}A$, $C = \frac{r+1}{p+1}B$, $D = \frac{r+2}{p+2}C$, $E = \frac{r+3}{p+3}D$, &c. Therefore there is the same relation between the areas of these curves and the terms of the proposed series, when the abscissa x is unity. *Q. E. D.*

COROLLARY.

HENCE in the series a, $\frac{r}{p}a$, $\frac{r+1}{p+1}b$, $\frac{r+2}{p+2}c$, &c. if z denote the distance

between

between the firſt term a, and any other primary or intermediate term T; it will be, as the area of a curve whoſe ordinate is $x^{r-1} \times \overline{1-x}\vert^{p-r-1}$, to the area of a curve whoſe ordinate is $x^{r+z-1} \times \overline{1-x}\vert^{p-r-1}$; ſo is a the firſt term, to any other primary or intermediate term, at the diſtance z from the beginning.

E X A M P L E I.

Let there be given this ſeries 1, $\frac{1}{4}a$, $\frac{3}{6}b$, $\frac{5}{6}c$, $\frac{7}{8}d$, &c. to be intercalated. Becauſe the common difference both of the numerators and denominators is 2, divide them by 2, that the difference may be made unity, as in the theorem, and the ſeries will be 1, $\frac{\frac{1}{2}}{1}a$, $\frac{\frac{3}{2}}{2}b$, &c. which compared with that in the propoſition, gives $p=1$, $r=\frac{1}{2}$, which being ſubſtituted, it will be as the area belonging to the ordinate $x^{-\frac{1}{2}} \times \overline{1-x}\vert^{-\frac{1}{2}}$ to the area belonging to the ordinate $x^{z-\frac{1}{2}} \times \overline{1-x}\vert^{-\frac{1}{2}}$; that is, as the area of $\frac{1}{\sqrt{x-xx}}$, to the area of $\frac{x^z}{\sqrt{x-xx}}$; ſo is the firſt term of the ſeries, or unity, to any other primary or intermediate term at the diſtance z from the beginning.

As if the term required ſtand in the middle between the firſt and ſecond, z will be $=\frac{1}{2}$, in which caſe the latter ordinate becomes $\frac{x^{\frac{1}{2}}}{\sqrt{x-xx}}$, or $\frac{1}{\sqrt{1-x}}$: Therefore as the area of the ordinate $\frac{1}{\sqrt{x-xx}}$ to the area of the ordinate $\frac{1}{\sqrt{1-x}}$, that is, as the circumference of a circle $3.1415926\ldots.$ &c. is to 2, ſo is unity to the term in the middle between the firſt and ſecond, which therefore is $.63661977\ldots.$ &c.

If the $101^{ſt}$ term of the ſame ſeries be required, z will be $=100$; therefore as $3.1415\ldots.$ &c. to the area of the ordinate $\frac{x^{100}}{\sqrt{x-xx}}$; ſo is unity to the propoſed term. And likewiſe by putting $z=100\frac{1}{2}$, the term in the middle between the hundred and firſt and hundred and ſecond will be determined by the circumference of a circle and area of the ordinate $\frac{x^{100}}{\sqrt{1-x}}$; by taking every where thoſe parts of the areas whoſe abſciſſas are each equal to unity.

Likewiſe may the reciprocals of the terms be interpolated; and that ſome-

sometimes more commodiously than the terms themselves. The reciprocals of the terms in the last series are 1, $\frac{1}{4}a$, $\frac{4}{7}b$, &c. Therefore $r=1$, $p=\frac{1}{4}$; whence as the area of the ordinate $x^0 \times \overline{1-x}|^{-\frac{1}{4}}$, is to the area of the ordinate $x^z \times \overline{1-x}|^{-\frac{1}{4}}$; so is the first term, to the term at the distance z from the beginning; that is, as 2 to the area of the ordinate $\dfrac{x^z}{1-x\sqrt{1-x}}$; for the first curve is quadrable.

E X A M P L E II.

SUPPOSE the series 1, $\frac{1}{4}a$, $\frac{4}{7}b$, $\frac{7}{10}c$, $\frac{10}{13}d$, &c. to be interpolated: divide the numerators and denominators by 3 their increment; and you will find $p=\frac{2}{3}$, $r=\frac{1}{3}$; hence the former ordinate becomes $x^{-\frac{2}{3}} \times \overline{1-x}|^{-\frac{2}{3}}$, or $\dfrac{1}{\sqrt[3]{x^2-2x^3+x^4}}$; and the latter $\dfrac{z^z}{\sqrt[3]{x^2-2x^3+x^4}}$: then as the area of the former ordinate is to the area of the latter, so is the first term of the series to any other whose distance from the beginning is z.

P R O P O S I T I O N XXV.

IF in the ordinate of a curve $x^{p-z} \times \overline{1-x}|^{r+z-1}$ be wrote successively the whole numbers 0, 1, 2, 3, 4, &c. for z, the relation between the areas of the ordinates coming out will be the same as between the terms of the series a, $\frac{r}{p}$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, &c. where the numerators continually increase, but the denominators decrease. And here I also put the abscissa x equal to unity.

This proposition is demonstrated after the same manner as the above.

C O R O L L A R Y.

HENCE in the series a, $\frac{r}{p}a$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, &c. as the first term a is to any other distant from the beginning by the space z, so is the area of a curve whose ordinate is $x^p \times \overline{1-x}|^{r-1}$ to the area of a curve whose ordinate is $x^{p-z} \times \overline{1-x}|^{r+z-1}$.

E X A M P L E I.

GIVEN this series 1, $\frac{n}{1}a$, $\frac{n-1}{2}b$, $\frac{n-2}{3}c$, $\frac{n-3}{4}d$, &c. to be interpolated, whose terms are the uncias of a binomial in the power whose index is n.

H h   Because

Becaufe this feries does not fall directly under this propofition, I interpo-
late the reciprocals of the terms $1, \frac{1}{n}a, \frac{2}{n-1}b, \frac{3}{n-2}c$, &c. which done, r
will be $=1, p=n$; therefore as the area of the ordinate $x^z \times \overline{1-x}|^\circ$ to the
area of the ordinate $x^{n-z} \times \overline{1-x}|^z$, fo is unity to the term diftant from the
beginning by the fpace z, in the latter feries; or, as the area of the ordinate
$x^{n-z} \times \overline{1-x}|^z$ to $\frac{1}{n+1}$, fo is unity to a term in the firft feries, at the di-
ftance z from the beginning.

As if the uncia of the fifth term in the ninth power be defired, n will be
$=9, z=4$; which being wrote, the area of the ordinate $x^5 \times \overline{1-x}|^4$ to $\frac{1}{10}$
as unity to the uncia fought.   But the ordinate evolved is $x^5-4x^6+6x^7$
$-4x^8+x^9$, and its area $\frac{1}{6}-\frac{4}{7}+\frac{6}{8}-\frac{4}{9}+\frac{1}{10}$, or $\frac{1}{1260}$; then as $\frac{1}{1260}$ to $\frac{1}{10}$,
fo is unity to 126, which is the uncia propofed.

EXAMPLE II.

IF in the fimple power of a binomial, there be fought the term which
ftands in the middle between the two uncias 1 and 1; the index of the
binomial will be $n=1, z=\frac{1}{2}$; and from thence, as the area of the ordi-
nate $x^{\frac{1}{2}} \times \overline{1-x}|^{\frac{1}{2}}$ is to $\frac{1}{2}$, that is, as the area of a circle is to the circum-
fcrib'd fquare, fo is unity to the term between the uncias 1 and 1.

SCHOLIUM.

WHEN the curves that are to be fquared are of many dimenfions, find
fome of their ordinates by a table of logarithms, from which will be given
the are as by *Newton*'s parabola. But if the relation of the terms in a feries to
be interpolated be among many terms, the interpolation will be perform'd
by the comparifon of other curves. But feting afide this, I fhall add fome-
thing concerning other methods of interpolation.

PROPOSITION XXVI.

SUPPOSE *the feries to be interpolated be* $1, \frac{r}{p}a, \frac{r+1}{p+1}b, \frac{r+2}{p+2}c, \frac{r+3}{p+3}$
d, &c. and n be put $=r-p$, *and*

$B=$

$$B = \frac{n}{1} \cdot \frac{n-1}{2} \text{ into A,}$$

$$C = \frac{n-1}{2} \cdot \frac{n-2}{2} \text{ into } B + \frac{n}{3}A,$$

$$D = \frac{n-2}{3} \cdot \frac{n-3}{2} \text{ into } C + \frac{n-1}{3}B + \frac{n}{3} \cdot \frac{n-1}{4}A,$$

$$E = \frac{n-3}{4} \cdot \frac{n-4}{2} \text{ into } D + \frac{n-2}{3}C + \frac{n-1}{3} \cdot \frac{n-2}{4}B + \frac{n}{3} \cdot \frac{n-1}{4} \cdot \frac{n-2}{5}A,$$

$$F = \frac{n-4}{5} \cdot \frac{n-5}{2} \text{ into } E + \frac{n-3}{3}D + \frac{n-2}{3} \cdot \frac{n-3}{4}C + \frac{n-1}{3} \cdot \frac{n-2}{4} \cdot \frac{n-3}{5}B$$
$$+ \frac{n}{3} \cdot \frac{n-1}{4} \cdot \frac{n-2}{5} \cdot \frac{n-3}{6}A;$$
&c.

then will z^n *into* $A + \dfrac{B}{z} + \dfrac{C}{z^2} + \dfrac{D}{z^3} + \dfrac{E}{z^4} +$ &c. *be the primary or intermediate term of the series to be interpolated, whose distance from the beginning is* $z - p$.

It is to be noted that the coefficient A, by the eighteenth Proposition, is equal to the term in the series of the numerators $1, ra, \overline{r+1}.b, \overline{r+2}.c$, &c. distant from the beginning by the space $p - r$; and that it is determined by the second example of the twenty first Proposition.

D E M O N S T R A T I O N.

The series proposed is defined by this differential equation $T' = \dfrac{z+n}{z}$ T, where $n = r - p$, as in the Theorem, and the successive values of the indeterminate quantity z are p, $p + 1$, $p + 2$, &c. Suppose now

$$T = z^n \text{ into } A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \frac{E}{z^4} + \text{ &c.}$$

then for T and z write their succeeding values T' and $z + 1$ respectively, and it will be

$$T' = \overline{z+1}|^n \text{ into } A + \frac{B}{z+1} + \frac{C}{\overline{z+1}|^2} + \frac{D}{\overline{z+1}|^3} + \frac{E}{\overline{z+1}|^4} + \text{ &c.}$$

By evolving the powers, $T' = z^n$ into $A +$

$$\frac{nA+B}{z} + \frac{nA + 2B \cdot \overline{n-1} + 2C}{2z^2} + \frac{nA + 3B \cdot \overline{n-1} \cdot \overline{n-2} + 6 \cdot \overline{n-2} \cdot C + 6D}{6z^3} +$$
&c.

Let

Let the equation $T' = \frac{z+n}{z}T$, to be refolved be written thus, $T'z - Tz - Tn = 0$; and in the fame fubftitute the values of T and T', and there will refult

$$z^n \text{ into } \frac{n.\overline{n-1}A - 2B}{2z} + \frac{\overline{nA+_3B}.\overline{n-1}.\overline{n-2} - 12C}{6z^2} + \&c. = 0.$$

Let now the numerators be put equal to nothing, and we fhall have $B = \frac{n}{1}.\frac{n-1}{2}$ into A, $C = \frac{n-1}{2}.\frac{n-2}{2}$ into $B + \frac{n}{3}A$, &c. And by making a computation, we fhall get other coefficients as in the Theorem. *Q. E. D.*

EXAMPLE I.

Let this feries 1, $\frac{1}{2}a$, $\frac{1}{4}b$, $\frac{1}{6}c$, $\frac{1}{8}d$, &c. be propofed, which is defined by the equation $T' = \frac{z - \frac{1}{2}}{z}T$, in which the fucceffive values of z are 1, 2, 3, 4, &c. thefe compared with the equation $T' = \frac{z+n}{z}T$, gives $n = -\frac{1}{2}$, and from thence

$B = \frac{1}{1}$ into A	$= A \times \frac{1}{8}$,
$C = 1\frac{1}{16}$ into $B - \frac{1}{6}A$	$= A \times \frac{25}{128}$,
$D = 1\frac{1}{24}$ into $C - \frac{2}{6}B + \frac{1}{18}A$	$= A \times \frac{105}{1024}$,
$E = \frac{6\frac{1}{2}}{13}$ into $D - \frac{1}{6}C + \frac{1}{18}B - \frac{1}{72}A$	$= A \times \frac{1659}{32768}$,
$F = \frac{9\frac{9}{2}}{20}$ into $E - \frac{1}{6}D + \frac{11}{72}C - \frac{7}{72}B + \frac{7}{144}A$	$= A \times \frac{6237}{262144}$,
&c.	

Therefore

$$T = \frac{A}{\sqrt{z}} \text{ into } 1 + \frac{3}{8z} + \frac{25}{128z^2} + \frac{105}{1024z^3} + \frac{1659}{32768z^4} + \frac{6237}{262144z^5} + \&c.$$

But the quantity A in fuch like examples may be determined thus: by the relation of the terms to be interpolated, feek any primary one far enough diftant from the beginning, for inftance, the fixteenth, which is $.144464448....$ &c. Write this for T, and in the mean time for z its correfpondent value, namely 16, and you will have

$$.144464448 = \frac{A}{4} \text{ into } 1 + \frac{3}{8.16} + \frac{25}{128.16.16} + \&c.$$

or, by gathering the terms into one fum, $.144464448 = \frac{A}{4}$ into 1.02422627, whence there comes out $A = .564189583548$; which being given, T will be given in any other cafe by a very few terms of its value.

EX-

E X A M P L E II.

LET this series 1, $\frac{2}{1}a$, $\frac{4}{4}b$, $\frac{6}{7}c$, $\frac{11}{10}d$, &c. which is defined by the equation $T'=\frac{z+\frac{1}{3}}{z}T$, $\frac{4}{1}$, $\frac{4}{3}$, $\frac{7}{5}$, $\frac{10}{7}$, &c. being the succeſſive values of the abſciſſa z: this compared with the equation in the theorem, gives $n=\frac{1}{3}$, which being ſubſtituted, we ſhall have

$$T=A\sqrt[3]{z} \text{ into } 1-\frac{1}{9z}*+\frac{10}{2187z^3}+\frac{11}{19683z^4}-\frac{77}{59049z^5}+ \&c.$$

Now to find the coefficient A, I ſeek the fourteenth term of the ſeries to be interpolated, which is 4.652136; then I write this value for T, and for z its fourteenth value $\frac{40}{3}$, and I have

$$4.652136=A\sqrt[3]{\frac{40}{3}} \text{ into } 1-\frac{1}{120}+\frac{1}{311400}+ \&c.$$

Or, by extracting the cube root of $\frac{40}{3}$, and by collecting the terms into one ſum, I get 4.652136=A×2.351506; therefore A=1.978364. A being now given, any other term will be found with the greateſt eaſe. Required that which ſtands a third part of the common diſtance before the 1001ſt term; for z write 1000, its correſpondent value, and the value of T will be 10A into $1-\frac{1}{9000}$, or T=19.78144. For when the deſired term is a long way diſtant from the beginning, and the computation is not to be produced to many figures, a very few terms in the value of T do abundantly ſuffice.

S C H O L I U M.

IN the ſame manner as in this propoſition the root is extracted from the equation $T'=\frac{z+n}{z}T$; likewiſe it is extracted from any other which is contained under this form

$$T\times z^\theta \overline{+az^{\theta-1}+bz^{\theta-2}}+ \&c. =T'\times z^\theta \overline{+cz^{\theta-1}+dz^{\theta-2}}+ \&c.$$

For the index n is $=a-c$ by the ſixth Propoſition; and from thence the form of the ſeries to be aſſumed for T will be

$$T=Az^n+Bz^{n-1}+Cz^{n-2}+ \&c.$$

But ſince in ſuch like ſeries, which are the roots of differential equations, the indices of z have unity for the decrement, except in ſome very particular caſes, therefore having the index of z in the firſt term, the form of the ſeries to be aſſumed for the root T is had, then by writing $z+1$ for z, and T' for T, there will come out

$$T'=A.\overline{z+1}^n+B.\overline{z+1}^{n-1}+C.\overline{z+1}^{n-2}+ \&c.$$

I i   Then

Then by reducing this value to the form of T, as was shewn above, and by drawing each value into the quantities which the equation to be resolv'd requires; the assumed coefficients will be given from a comparison of the homologous terms in the resulting equation.

PROPOSITION XXVII.

IF an equation to the series be $T' = \dfrac{zz}{zz+r}T$, the root will be

$$T = A + \frac{r}{z}A + \frac{r+1}{2.z+1}B + \frac{r+4}{3.z+2}C + \frac{r+9}{4.z+3}D + \frac{r+16}{5.z+4}E + \&c.$$

Let the equation $T' = \dfrac{zz}{zz+r}T$ to be resolv'd be written thus, zzT $zzT' - rT' = 0$, and assume

$$T = A + \frac{B}{z} + \frac{C}{z.z+1} + \frac{D}{z.z+1.z+2} + \frac{E}{z.z+1.z+2.z+3} + \&c.$$

then will

$$T^v = A + \frac{B}{z+1} + \frac{C}{z+1.z+2} + \frac{D}{z+1.z+2.z+3} + \frac{E}{z+1.z+2.z+3.z+4} + \&c.$$

whence $T - T' = \dfrac{B}{z.z+1} + \dfrac{2C}{z.z+1.z+2} + \dfrac{3D}{z.z+1.z+2.z+3} +$

$$\frac{4E}{z.z+1.z+2.z+3.z+4} + \&c.$$

and by drawing it into zz

$$zzT - zzT' = \frac{Bz}{z+1} + \frac{2Cz}{z+1.z+2} + \frac{3Dz}{z+1.z+2.z+3} +$$

$$\frac{4Ez}{z+1.z+2.z+3.z+4} + \&c.$$

and by reduction

$$zzT - zzT' = B + \frac{2C-B}{z+1} + \frac{3D-4C}{z+1.z+2} + \frac{4E-9D}{z+1.z+2.z+3} + \&c.$$

Write this value for $zzT - zzT'$, and for T^v its value now found, and there will result

$$B - rA + \frac{2C - \overline{r+1}B}{z+1} + \frac{3E - \overline{r+4}C}{z+1.z+2} + \frac{4E - \overline{r+9}D}{z+1.z+2.z+3} + \&c. = 0.$$

And by comparing the homologous terms $B = \dfrac{r}{1}A$, $C = \dfrac{r+1}{2}B$, $D = \dfrac{r+4}{3}C$, $E =$

C, E$=\frac{r+9}{4}$D, &c.   Thefe are the values of the coefficients; but if A, B, C, &c. denote the whole terms, the value of T will then come out as here affign'd. *Q. E. D.*

E X A M P L E.

Dr *Wallis* found, that the laft term of this feries 1, $\frac{1}{9}$A, $\frac{24}{25}$B, $\frac{45}{49}$C, $\frac{70}{81}$D, &c. was the area of a circle whofe diameter is unity; where the denominators are the fquares of the odd numbers, and greater than the numerators by unity. Now let us fee here what the laft term of this feries 1, $\frac{4}{3}$A, $\frac{16}{15}$B, $\frac{36}{35}$C, $\frac{64}{63}$D, &c. is, where the numerators are fquares of the even numbers, and greater than the denominators by unity.   The equation to this feries will be T$'=\frac{zz}{zz-\frac{1}{4}}$T, 1, 2, 3, 4, &c. being the fucceffive values of the abfciffa z; therefore by comparing this equation with that in the propofition, it will be $r=-\frac{1}{4}$, which being fubftituted, there comes out

$$T=A-\frac{A}{4z}+\frac{3B}{8.z+1}+\frac{15C}{12.z+2}+\frac{35D}{16.z+3}+\frac{36E}{20.z+4}+ \text{\&c.}$$

And to determine the coefficient A, feek the tenth term of the feries, namely 1.5300.1727.35, which fubftitute for T, and 10 for z its cor-refpondent value; and you will obtain

$$1.5300.1727.35=A \text{ into } 1-\frac{1}{4.10}-\frac{1.3}{4.8.10.11}-\frac{1.3.15}{4.8.12.10.11.12} \text{ \&c.}$$

that is, by collecting the terms into one, 1.5300.1727.35 $=$ A into .9740.392454; and from thence A $=$ 1.57079633, the femi-circumference of a circle: which being given, any primary or intermediate term of the interpolated feries will very eafily be given; but it appears that the laft term thereof, or the product under all thefe $\frac{4}{3} \times \frac{16}{15} \times \frac{16}{15} \times \frac{64}{63} \times \frac{100}{99} \times$ &c. is equal to the firft coefficient A, confequently to the femiperiphery of a circle.

P R O P O S I T I O N XXVIII.

TO find the fum of any number of logarithms, whofe numbers are in a-rithmetical progreffion.

Let $x+n$, $x+3n$, $x+5n$, $x+7n \ldots z-n$, denote fo many numbers in arithmetical progreffion, whofe firft is $x+n$, laft $z-n$, and common difference $2n$. Moreover let l, z, and l, x denote the tabular logarithms of

of z and x; and let $a = .43429.44819.03252$, namely to the reciprocal of the natural logarithm of 10; and the sum of the proposed logarithms will be equal to the difference between the two following series:

$$\frac{zl, z}{2n} - \frac{az}{2n} - \frac{an}{12z} + \frac{7an^3}{360z^3} - \frac{31an^5}{1260z^5} + \frac{127an^7}{1680z^7} - \frac{511an^9}{1188z^9} + \&c.$$

$$\frac{xl, x}{2n} - \frac{ax}{2n} - \frac{an}{12x} + \frac{7an^3}{360x^3} - \frac{31an^5}{1260x^5} + \frac{127an^7}{1680x^7} - \frac{511an^9}{1188x^9} + \&c.$$

And these series are thus continued *in infinitum*; put

$$-\frac{1}{3 \cdot 4} = A,$$

$$-\frac{1}{5 \cdot 8} = A + 3B,$$

$$-\frac{1}{7 \cdot 12} = A + 10B + 5C,$$

$$-\frac{1}{9 \cdot 16} = A + 21B + 35C + 7D,$$

$$-\frac{1}{11 \cdot 20} = A + 36B + 126C + 84D + 9E,$$

&c.

where the numbers which are multiplied into A, B, C, D, &c. in the different values, are the alternate uncias in the odd powers of a binomial. These being premised, the coefficient of the third term will be $-\frac{1}{12} = A$, that of the fourth $+\frac{7}{360} = B$, of the fifth $-\frac{31}{1260} = C$, &c.

DEMONSTRATION.

LET the variable quantity z be diminished by its decrement $2n$, or which is the same, substitute $z - 2n$ for z in the series.

$$\frac{zl, z}{2n} - \frac{az}{2n} - \frac{an}{12z} + \frac{7an^3}{360z^3} - \frac{31an^5}{1260z^5} + \&c.$$

and the successive value of the same will come out

$$\frac{\overline{z - 2nl, z - 2n}}{2n} - \frac{a}{2n} \times \overline{z - 2n} - \frac{an}{12 . z - 2n} + \frac{7an^3}{360 . \overline{z - 2n}^3} - \frac{31an^5}{1260 . \overline{z - 2n}^5} + \&c.$$

Subtract this from the former value, the terms first being reduced by division to the same form, and there will remain

$$lz - \frac{an}{z} - \frac{ann}{2z^2} - \frac{an^3}{3z^3} - \frac{an^4}{4z^4} - \&c.$$

that

that is, the logarithm of $z-n$. Therefore, universally, the decrement of two succeſsive values of the series is equal to the logarithm of $z-n$, which expreſses in general any of the logarithms that were to be ſummed. Therefore the series will be the ſum of the propoſed logarithms, if the other ſeries be taken from it. For the ſums, as well as the areas, muſt ſometimes be corrected, that the true ones may come out.

E X A M P L E I.

LET it be propoſed to find the ſum of the logarithms of the ten numbers, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119; theſe compared with $x+n$, $x+3n$, $x+58$ .... $z-n$, give the common difference $2n=2$, and $n=1$; and the firſt $x+1=101$, the laſt $z-1=119$; whence $x=100$, $z=120$. And theſe being ſubſtituted, and $.43429.44819.03252$ for a; and the logarithms of 100 and 120 reſpectively for l, x and l, z; the values of the two ſeries will be found to be $78.28491.40012.1$, and $98.69290.42601.6$, whoſe differences give $20.40799.02589.5$ for the required ſum of the logarithms.

E X A M P L E II.

REQUIRED the ſum of the logarithms of the numbers 11, 12, 13, ... 1000, whoſe firſt is 11, and laſt 1000, and common difference unity. Therefore n is $=\frac{1}{2}$, $x+\frac{1}{2}=11$, $z-\frac{1}{2}=1000$; whence $x=\frac{21}{2}$, $z=\frac{2001}{2}$; which being wrote, and the logarithms of $\frac{21}{2}$ and $\frac{2001}{2}$ for l, x and l, z, there will come out for the values of the ſeries $2567.20555.42879$, and $6.16067.30987$, whoſe difference leaves $2561.04488.11892$ for the ſum of the logarithms ſought.

But if you will have the ſum of any number of the logarithms of the natural numbers 1, 2, 3, 4, 5, &c. put $z-n$ to be the laſt of the numbers, n being $=\frac{1}{2}$; and three or four terms of this ſeries zl, $z-az-$ $\frac{a}{24z}+\frac{7a}{2880z^3}-$ &c. added to half the logarithm of the circumference of a circle whoſe radius is unity, that is, to $0.39908.99341.79$, will give the ſum ſought; and that with leſs labour the more logarithms are to be ſummed. Thus if you put $z-\frac{1}{2}=1000$, or $z=\frac{2001}{2}$, the value of the ſeries will be $2567.20555.42879$, as before; which added to the conſtant logarithm makes $2567.60464.42221$ for the ſum of the logarithms of a thouſand of the firſt numbers of this ſeries 1, 2, 3, 4, 5, &c.

E X A M P L E III.

REQUIRED to find the five hundredth uncia in the thouſandth power of a binomial.

It

It is manifest, by *Newton*'s theorem for evolving a binomial, that the uncia is equal to the product under four hundred and ninety nine factors $\frac{1000}{1}$, $\frac{999}{2}$, $\frac{998}{3}$, $\frac{997}{4}$, $\frac{996}{5}$, ... $\frac{502}{499}$, whose first is $\frac{1000}{1}$, and last $\frac{502}{499}$, both the numerators and denominators being in arithmetical progression. To find the sum of the logarithms of the numerators 1000, 1999, 998, 997, ... 502 ; put the common difference $1 = 2n$, their greatest 1000 $= z - \frac{1}{2}$, the least $502 = x + \frac{1}{2}$; and n will be $= \frac{1}{2}$, $z = 1000\frac{1}{2}$, $x = 501\frac{1}{2}$, which being substituted, there will come out $2567.20555.42879$ for the value of the first series, and $1136.38715.63268$ for the value of the latter ; but their difference $1430.81839.79611$ is equal to the sum of the logarithms of the numerators. Then to obtain the sum of the logarithms of the denominators 1, 2, 3, 4 .... 499 ; put $n = \frac{1}{2}$, $z - \frac{1}{2} = 499$, or $z = 499\frac{1}{2}$; and these being wrote in the first series, there will come out its value $1130.98834.85966$, to which add the logarithm $.39908.99342$, according to the rule laid down in the former example ; and you will have $1131.38743.85308$ for the sum of the logarithms of the denominators ; which lastly taken from the sum of the logarithms of the numerators, and there will remain $299.43095.94303$ the logarithm of the required uncia.

S C H O L I U M.

A SERIES of this sort, 1, $\frac{r}{p}A$, $\frac{r+1}{p+1}B$, $\frac{r+2}{p+2}C$, &c. is interpolated by the twenty sixth Proposition, when the difference between r and p is small ; and generally, by this Proposition, without any regard to its difference. And in the same manner I can find the sum of the logarithms of numbers, which are more compounded by far than the equi-different ones ; and by that means to assign the terms of the series whose intercalation uses to be reckoned very difficult. By this Problem are also found the areas of curves whose ordinates are of this nature $\overline{1-x}\vert^{1000}$, where the index of the binomial is very great ; but in that case only when the part of the area sought stands upon a part of the abscissa equal to unity.

And indeed almost all problems concerning interpolations come under this analysis, nay even when three or more terms of the series to be intercalated are in the differential equation ; for the resolution of these I have in my power. And it may be proper to observe that the series which come out by *Newton*'s parabola, come out likewise by our method. For let the intercalation of this series

$$a, \quad \frac{r+n}{r}a, \quad \frac{r+n+1}{r+1}b, \quad \frac{r+n+2}{r+2}c, \quad \frac{r+n+3}{r+3}d, \quad \&c.$$

be

be proposed, which is defined by the equation $T' = \frac{z+n+r}{z+r}T$, in which the succeſſive values of the abſciſſa z are 0, 1, 2, 3, &c. and let it be wrote after the following manner, $\overline{z+r} \times \overline{T'-T} - nT = 0$; and ſuppoſe

$$T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

And by ſubſtituting T' for T, and $z+1$ for z, we ſhall get

$$T' = A + B\overline{z+1}. + C\overline{z+1}.z + D\overline{z+1}.z.\overline{z-1} + E\overline{z+1}.z.\overline{z-1}.\overline{z-2} + \&c.$$

whence $T' - T = B + 2Cz + 3Dz.\overline{z-1} + 4Ez.\overline{z-1}.\overline{z-2} + \&c.$

And theſe values being ſubſtituted in the equation to be reſolved, and the terms reduced to the ſame form, there will reſult.

$$\begin{Bmatrix} +rB \\ -nA \end{Bmatrix} + 2.\begin{Bmatrix} \overline{r+1}C \\ -\overline{n-1}B \end{Bmatrix} z + 3\begin{Bmatrix} \overline{r+2}D \\ -\overline{n-2}C \end{Bmatrix} z.\overline{z-1} + 4\begin{Bmatrix} \overline{r+3}E \\ -\overline{n-3}D \end{Bmatrix} z.\overline{z-1}.\overline{z-2} + \&c.$$

Laſtly, by putting the homologous terms equal to nothing, we ſhall have

$$B = \frac{n}{r}A, \quad C = \tfrac{1}{2} \times \frac{n-1}{r+1}B, \quad D = \tfrac{1}{3} \times \frac{n-2}{r+2}C, \quad E = \tfrac{1}{4} \times \frac{n-3}{r+3}D, \&c.$$

Therefore

$$T = A + A \times \frac{n}{r} \times \frac{z}{2} \times A \times \frac{n}{r} \times \frac{z}{1} \times \frac{n-1}{r+1} \times \frac{z-1}{r+2} + \&c.$$

That is,

$$T = A + \frac{n}{r}A\frac{z}{1} + \frac{n-1}{r+1}B\frac{z-1}{2} + \frac{n-2}{r+2}C\frac{z-2}{3} + \frac{n-3}{r+3}D\frac{z-3}{4}, \&c.$$

where A, B, C, D, &c. do not any more denote the coefficients, but the whole terms. And the firſt coefficient A, which is not determined from the equation, is equal to the term of the ſeries to be interpolated, which paſſes through the beginning of the abſciſſa z. And this is the very ſame value of T which had been produced by the nineteenth propoſition. And likewiſe the ſeries which the twentieth Propoſition exhibits, will be found by the reſolution of differential equations, by aſſuming a due form of the root.

P R O P O S I T I O N   XXIX.

GIVEN *a ſeries of ordinates, at any diſtances from one another, pro-ceeding one way only in infinitum; required to find a parabolic line, that ſhall paſs through the extremities of them all.*

Let A, A1, A2, A3, A4, &c. be ordinates ſtanding on the abſciſſa at right angles, and let R be any point in the abſciſſa; and put $a = $ R A, $b = $ R A1, $c = $ R A2, $d = $ R A3, $e = $ R A4, &c. namely let a, b, c, d, e, &c. be the diſtances between the ordinates and the point R reſpectively, and let T denote any ordinate in general whoſe diſtance from the point R is z, then put
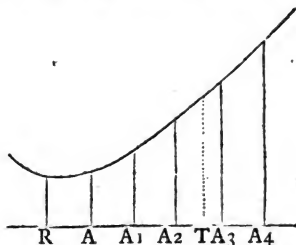
$$B = \frac{A_1 - A}{b - a}, \quad C = \frac{B_1 - B}{c - a}, \quad D = \frac{C_1 - C}{d - a}, \quad E = \frac{D_1 - D}{e - a}, \&c.$$

$$B_1 = \frac{A_2 - A_1}{c - b}, \quad C_1 = \frac{B_2 - B_1}{d - b}, \quad D_1 = \frac{C_2 - C_1}{e - b}, \&c.$$

$$B_2 = \frac{A_3 - A_2}{d - c}, \quad C_2 = \frac{B_3 - B_2}{e - c}, \&c.$$

$$B_3 = \frac{A_4 - A_3}{e - d}, \&c.$$

&c.

And the ordinate will be $T = A +$
$$B \times \overline{z - a} +$$
$$C \times \overline{z - a} \times \overline{z - b} +$$
$$D \times \overline{z - a} \times \overline{z - b} \times \overline{z - c} +$$
$$E \times \overline{z - a} \times \overline{z - b} \times \overline{z - c} \times \overline{z - d} +$$
$$F \times \overline{z - a} \times \overline{z - b} \times \overline{z - c} \times \overline{z - d} \times \overline{z - e} +$$
&c.

It is to be obferved that the beginning of the abfcifla, namely the point R, is taken at pleafure, either within or without all the ordinates, as in the fcheme, having a due regard to the figns $+$ and $-$. And the propofition is demonftrated by fubftituting the ordinates A, A1, A2, &c. fucceffively for T, and the lengths a, b, c, &c. fucceffively for z. For by taking the differences of the equations coming out, and dividing them by the intervals of the ordinates, the values of the coefficients will come out as above affigned.

E X A M P L E   I.

LET the diftance of the ordinates from the beginning of the abfcifla be $a = 2$, $b = 3$, $c = 5$, $d = 6$, and the ordinates, $A = 2$,

$$A = 2, \quad B = 1, \quad C = 0, \quad D = \tfrac{1}{4},$$
$$A_1 = 3, \quad B_1 = 1, \quad C_1 = 2,$$
$$A_2 = 5, \quad B_2 = 7,$$
$$A_3 = 12,$$

and by making a computation according to what is laid down in the theorem, there will be found $B = 1$, $C = 0$, $D = \tfrac{1}{4}$, which being substituted, and 2 for A, we shall find $T = 2 + \overline{z-2} + \tfrac{1}{4} \times \overline{z-2}.\overline{z-3}.\overline{z-5}$; which reduc'd into order, becomes $T = \dfrac{z^3 - 10z^2 + 33z - 30}{2}$. For if there be wrote 2, 3, 5, 6 for z, there will come out 2, 3, 5, 12, the ordinates proposed.

EXAMPLE II.

REQUIRED to determine the time of the solstice from a few given meridian altitudes of the sun about the same time.

Let ordinates denote the sun's altitude, and their distances denote the times between the observations ; then let a parabola pass through the extremities of the ordinates, and its abscissa which corresponds to the least ordinate, whether it be one of the given ordinates, or any intermediate one, will determine the moment of time that the sun enters the tropic. For *example*, B. *Walterus* in the year 1500 observed at *Noremburg* the distances of the sun from the zenith, as follows,

$$\left. \begin{matrix} 44975 & 8^{th} \\ 44934 & 9 \\ 44883 & 12 \\ 44990 & 16 \end{matrix} \right\} \text{ of } \textit{June.}$$

Let now the distance observ'd on the 8th day be the first ordinate, and therein the beginning of the abscissa, and a will be $= 0$, $b = 1$, $c = 4$, $d = 8$, and the computation will be

$$A = 44975, \quad B = -41, \quad C = +6, \quad D = +\tfrac{1}{11},$$
$$A_1 = 44934, \quad B_1 = -17, \quad C_1 = +\tfrac{175}{11},$$
$$A_2 = 44883, \quad B_2 = +\tfrac{107}{4},$$
$$A_3 = 44990,$$

And by substituting these values for A, B, C, D, there will be found $T = 44975 - 41z + 6z.\overline{z-1} + \tfrac{1}{11}z.\overline{z-1}.\overline{z-4}$, that is, $T = 44975 - \tfrac{171}{6}z + \tfrac{187}{11}z^2 + \tfrac{1}{11}z^3$. Now because the abscissa sought corresponds to the least ordinate, put the Fluxion of $T = 0$, and we shall have $3zz + 374z = 1500$, whose root 3.889355 expresses the days elaps'd between the noon of the 8th day of *June* and the moment of the solstice, which

L l   there

therefore happens 21^h $20^m.\frac{1}{2}$ nearly paſt the noon of the 11th day, according to theſe obſervations. Likewiſe may the time of the ſolſtice be determined by more obſervations, and a parabola of more dimenſions ; or by three obſervations, by uſing a conic parabola, as Dr *Halley* hath ſhewn ; but it requires the differences between the obſerved altitudes, to be ſenſibly greater than the errors which may be committed in the obſervations, otherwiſe nothing certain will be concluded.

SCHOLIUM.

Sir *Iſaac Newton* made uſe of this Propoſition to determine the place of a comet, which falls among ſome places known by obſervations ; namely, if there be obſerved as many longitudes as you pleaſe, denoted by ſo many ordinates, whoſe diſtances are proportional to the times between the obſervations, and a parabola be deſcribed through the extremities of the ordinates, the intermediate ordinates of this figure will denote the intermediate longitudes of the comet for the times that are proportional to their abſciſſas. And by the ſame method will be given the latitude for any time from certain given latitudes ; and from the longitude and latitude being given, there is given the place of the comet in the heavens. And after this manner many things, which are difficult to obſerve at a certain time, may be determined, accurately enough, from certain obſervations taken before and after that time.

Likewiſe this problem is applicable to the reſolution of pure or adfected equations. For in an equation to be reſolved, by writing for the root numbers not differing much from it, there will come out their diſtances, which being interpolated will exhibit the root. But, after Dr *Halley*'s method of the reſolution of equations, a more compendious one is not to be expected.

In that caſe, when the diſtances of the ordinates are diminiſhed *in infinitum*, this Problem will give the root of a fluxional equation, tho' neither the root nor any other indeterminate quantity flow uniformly ; and this by a meer ſubſtitution of the Fluxions of the root for the differences of the ordinates, and for their diſtances the Fluxions of the abſciſſa. For as the caſes of the equidiſtant ordinates anſwer to the Fluxions of the abſciſſa uniformly increaſing, ſo this Propoſition anſwers to Fluxions varying by any law. And the reſolution of a fluxional equation, wherein each indeterminate quantity flows by any law, is not a Corollary from this Propoſition, but a caſe thereof the moſt ſimple of any ; which, by the way, I thought proper here to take notice of, that it may appear that the differential method comprizes in general the univerſal doctrine of ſeries, which perhaps others did not perceive.

<div align="right">PRO-</div>

PROPOSITION XXX.

TO *find the asymptote of an hyperbola of the logarithmic kind, from some given equidistant ordinates thereof.*

Let the equidistant ordinates be A, B, C, D, E, &c. standing upon the abscissa PL at right angles; and let QR, the asymptote to the curve, be parallel to the abscissa, and distant from it by the space PQ. Let any abscissa AL be called z; and Lm, the correspondent ordinate, y; and let the logarithmic hyperbola HmK be defined by an equation of this form,

$$y = a - br^z - cr^{2z} - dr^{3z} - er^{4z} - \&c.$$
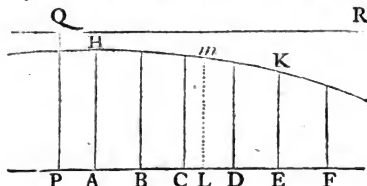
whence r, a, b, c, d, e, &c. are invariable quantities, and the distance between the abscissa and asymptote will be, *viz.*

$$PQ = A +$$
$$\frac{A - B}{r - 1} +$$
$$\frac{rA - \overline{r + 1}B + C}{r - 1 . r^2 - 1} +$$
$$\frac{r^3A - \overline{r^2 + r^2 + r}B + \overline{r^2 + r + 1}C - D}{r - 1 . r^2 - 1 . r^3 - 1} +$$
$$\frac{r^6A - \overline{r^5 + r^4 + r^3 + r^3}B + \overline{r^5 + r^4 + 2r^3 + r^2 + r}C - \overline{r^3 + r^2 + r + 1}D + E}{r - 1 . r^2 - 1 . r^3 - 1 . r^4 - 1} +$$

&c.

The coefficients of A, B, C, D, &c. in the different terms are formed by the continual multiplication of the numbers 1, $\frac{n-1}{r-1}$, $\frac{n-r}{r^2-1}$, $\frac{n-r^2}{r^3-1}$, &c. In the first term $n = r^0$, in the second $n = r^1$, in the third $n = r^2$, in the fourth $n = r^3$, &c. For *Example*, in the fourth term the coefficients taken in the inverse order are 1, $1 + r + r^2$, $r + r^2 + r^3$, r^3, neglecting the

the figns ; and $1 \times \dfrac{r^2-1}{r-1}$ is $= 1 + r + r^2$, $\overline{1 + r + r^2} \times \dfrac{r^3 - r}{r^2 - 1} = r + r^2 + r^3$;

laftly, $\overline{r + r^2 + r^3} \times \dfrac{r^3 - r^3}{r^3 - 1} = r^3$; fo therefore are the coefficients found. But in this Propofition I except the cafe wherein r is $= \pm 1$; for then the hyperbola degenerates into a right line.

And thus is the feries invefiigated : In the affumed equation $y = a - b r^z - c r^{2z} -$ &c. change the fign of the abfciffa z, and there will come out $y = a - \dfrac{b}{r^z} - \dfrac{c}{r^{2z}} - \dfrac{d}{r^{3z}} -$ &c. Let z now be infinitely great, and in the value of the ordinate y, all the terms will vanifh, except the firft, when r is conceived to be greater than unity ; and thereby y will be $= a$; that is, the ordinate removed to an infinite diftance, or the diftance between the abfciffa and afymptote PQ, is equal to a the firft term ; for in an infinite diftance the curve coincides with its afymptote. And the quantity a is thus invefiigated : In the equation before affum'd $y = a - b r^z - c r^{2z} - d r^{3z} - e r^{4z} -$ &c. write the equidiftant ordinates A, B, C, D, E, &c. fuccef-fively for y ; and alfo 0, 1, 2, 3, 4, &c. for the abfciffa z ; and there will come out thefe equations,

$$A = a - b - c - d - e - \text{&c.}$$
$$B = a - b r - c r^2 - d r^3 - e r^4 - \text{&c.}$$
$$C = a - b r^2 - c r^4 - d r^6 - e r^8 - \text{&c.}$$
$$D = a - b r^3 - c r^6 - d r^9 - e r^{12} - \text{&c.}$$
$$E = a - b r^4 - c r^8 - d r^{12} - e r^{16} - \text{&c.}$$
$$\text{&c.}$$

There are therefore as many equations as there are unknown quantities a, b, c, d, e, &c. from which feek a by common algebra, and its value will come out the fame as has been already affign'd for PQ. *Q. E. D.*

COROLLARY.

HENCE if fome initial terms be given in an infinite feries, whofe dif-ferences are nearly in geometrical progreffion, the laft will be given, be-ing that which is removed to an infinite diftance from the beginning. For if A, B, C, D, &c. taken in inverfe order, denote the terms whofe differences are nearly in geometrical proportion, as r^0, r^1, r^2, r^3, &c. the laft will be equal to PQ the diftance between the abfciffa and afymptote, or, in the ftyle of *James Gregory*, the termination of the feries will be given.

EX-

E X A M P L E.

GIVEN any regular polygons inscribed in a circle, to find the last of the polygons, or area of the circle.   Suppose they are

4	2.00000.00000.0000 =F	3.14033.11569.5475
8	2.82842.71247.4619 =E	126.08888.0294
16	3.06146.74589.2072 =D	6072.7439
32	3.12144.51522.5805 =C	5.5652
64	3.13654.84905.4594 =B	119
128	3.14033.11569.5475 =A	3.14159.26535.8979

Now call the last of the polygons A, the last but one B, the last but two C, and so backwards; and because their differences A—B, B—C, C—D, &c. are very nearly as the terms 1, 4, 16, 64, 256, &c. that is, as the powers of four, r will be =4; which being substituted, the general series becomes

$$A + \frac{A-B}{3} + \frac{4A - 5B + C}{3 \cdot 15} + \frac{64A - 84B + 21C - D}{3 \cdot 15 \cdot 63} + \&c.$$

In which write for A, B, C, &c. their values, and the first five terms will give the area of a circle to fifteen places of figures, as is manifest from the annex'd computation; and by a like method the thing is effected by circumscrib'd polygons.

And by this means any series may be summ'd; for if the equidistant ordinates denote the successive sums, the value of the whole series will be equal to the distance between the asymptote and abscissa. If a series to be summed be of this nature, $a + bx + cx^2 + dx^3 +$ &c. when x, x^2, x^3, &c. denote the parts of the terms which are in geometrical progression, r will be =x; and the value of PQ will converge so much the faster as the successive sums are more distant from the beginning, which are denoted by A, B, C, D, &c. But when $r = \pm 1$, an hyperbola must be assumed which is defined by an equation of this nature $y = \frac{1}{z^n}$ into $a + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3} +$ &c. instead of the logarithmic hyperbola; and the index n will be determined from the nature of the series to be summed up.

It is to be observed that infinite series may as easily be summed by Newton's parabola, as by these hyperbolas. For if the ordinates, which in hyperbolas are equidistant, be constituted at certain distances, they will exhibit, by the assistance of the parabola, the same expressions for the values of the series.

The number of figures which are true in a polygon A, are doubled by

two

two terms of the feries, tripled by three, &c. So, in the prefent example, 314 are the three true figures in the polygon A; and from thence five terms give the area of a circle to fifteen places of figures. And fuch approximations as thefe *James Gregory* and M. *Huygens* formerly found out, the one tripled the true figures, and the other quadrupled, and quinttupled them, and even produced them without limits, as may be feen in the *Appendix to the true Quadrature of the Circle and Hyperbola*.

PROPOSITION XXXI.

TO find the area of any curve, as near as possible, from some given equidiftant ordinates thereof.

Through the extremities of the ordinates defcribe a parabolic figure, and its area, which is found by the known methods, will be equal to the area of the propofed curve nearly. *Q. E. I.*

SCHOLIUM.

For as much as it would be laborious always to have recourfe to the parabola, I have computed the following table, which exhibits the area of a curve directly, from fome given equidiftant ordinates thereof.

A TABLE of AREAS.

$$3 \left| \frac{A+4B}{6}R \right.$$

$$5 \left| \frac{7A+32B+12C}{90}R \right.$$

$$7 \left| \frac{41A+216B+27C+272D}{840}R \right.$$

$$9 \left| \frac{989A+5888B-928C+10496D-4540E}{28350}R \right.$$

A TABLE of CORRECTIONS.

$$3 \left| \frac{P-4A+6B}{180}R \right.$$

$$5 \left| \frac{P-6A+15B-20C}{470}R \right.$$

$$7 \left| \frac{P-8A+28B-56C+70D}{930}R \right.$$

$$9 \left| \frac{P-10A+45B-120C+210D-252E}{1600}R \right.$$

In these tables A is the sum of the first and last ordinate, B of the second and last but one, C of the third and last but two, and so on, until you come to the ordinate in the midst of all, which is represented by the last of the letters A, B, C, &c. R is the base upon which the area lies, or the part of the abscissa intercepted between the first and last ordinate. P is the sum of two ordinates, whereof one stands before the first, the other after the last, at distances equal to that of the other ordinates. And the number of ordinates, which here is odd, is denoted at the side of the tables. The expressions in the tables of areas are the areas contained between the base, curve, and the extream ordinates. And those in the table of corrections are of the same magnitude as the differences between the true areas and those produced by the table. Therefore if the first figure of the correction be found, and added when the correction is negative, or subducted when affirmative, we may safely conclude that the area, thus corrected, is true in that place of decimals in which the first figure of correction enter, but no further; wherefore by a table of corrections the area found is corrected, and at the same time the number of true figures is known.

E X A M P L E.

Suppose $\frac{1}{1+x}$ be an ordinate of an equilateral hyperbola, and there be sought the area thereof which lies upon the abscissa equal to unity. For x write successively $\frac{0}{8}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, $\frac{8}{8}$, and there will come out nine ordinates,

$$\tfrac{8}{8}, \tfrac{8}{9}, \tfrac{8}{10}, \tfrac{8}{11}, \tfrac{8}{12}, \tfrac{8}{13}, \tfrac{8}{14}, \tfrac{8}{15}, \tfrac{8}{16}.$$

Therefore A$=\frac{8}{8}+\frac{8}{16}=\frac{3}{2}$, B$=\frac{8}{9}+\frac{8}{15}=\frac{64}{45}$, C$=\frac{8}{10}+\frac{8}{14}=\frac{48}{35}$, D$=\frac{8}{11}$ $+\frac{8}{13}=\frac{192}{143}$, E$=\frac{8}{12}=\frac{2}{3}$; and these being substituted in the last expression of the areas, and unity for R, there arises the area .69314721. Then in the ordinate $\frac{1}{1+x}$ write $\frac{-1}{8}$ and $+\frac{9}{8}$ successively for x, and there will come out two ordinates $\frac{8}{7}$ and $\frac{8}{17}$, whereof the former stands before the first, and the latter after the last; consequently P$=\frac{8}{7}+\frac{8}{17}=\frac{192}{119}$; which substitute for P, and for A, B, C, D, E, their values, and the correction for nine ordinates will give $+$.00000003, which since it is affirmative, let it be subtracted from the value of the area before found, and there will remain .69314718 exact in the last figure.

I had computed these tables further, but the expressions for eleven or more ordinates are useless by reason of the immense greatness of the numeral coefficients; but if nine ordinates do not give the area sufficiently accurate, divide the base into two or more parts, and by this the area is divid-

ed

ed into as many parts; then if you seek each of them separately by nine ordinates, you will have the whole area as accurate as you will. But it is also sometimes convenient to seek part of the area by an infinite series, especially if the curve cuts the base at right angles. And these being premised, any area will be had sufficiently accurate by the table already annex'd.

But the areas of curves may be expressed not incommodiously by the differences of the equidistant ordinates, in the following manner.

A TABLE of Areas by the Differences of the Ordinates.

$$1 \mid A$$
$$3 \mid A + \tfrac{1}{6}B$$
$$5 \mid A + \tfrac{1}{3}B + \tfrac{7}{90}C$$
$$7 \mid A + \tfrac{1}{3}B + \tfrac{11}{60}C + \tfrac{41}{840}D$$
$$9 \mid A + \tfrac{5}{9}B + \tfrac{16}{45}C + \tfrac{92}{945}D + \tfrac{7}{1400}E$$
$$11 \mid A + \tfrac{15}{6}B + \tfrac{175}{36}C + \tfrac{1445}{1512}D + \tfrac{4043}{3150}E + \tfrac{94}{4725}F$$
$$13 \mid A + 6B + \tfrac{103}{10}C + \tfrac{13}{10}D + \tfrac{1131}{700}E + \tfrac{481}{1155}F + \tfrac{66}{7050}G.$$

In this table A is the ordinate in the midst of them all, B is the second difference of three ordinates in the midst, C is the fourth difference of five ordinates in the middle, and so on to the last of the letters A, B, C, D, E, F, G, which is the last difference of all the ordinates. As suppose there be five ordinates a, b, c, d, e; A will be $= c$, B $= b - 2c + d$, C $= a - 4b + 6c - 4d + e$; and so in the other cases; and the expressions drawn into the base of the curve, or part of the abscissa contained between the first and last ordinates, give the areas from the given number of the ordinates which is denoted at the side. It is to be observed that the last terms in the expressions for nine, eleven, and thirteen ordinates, is not true, but more simple than these, and sufficiently near the true ones. For the middle ordinate A, and the differences B, C, D, E, &c. constitute a converging series; and therefore it is not required that the coefficients of the last terms which enter the computation, be precisely accurate. And from the convergency of the series A, B, C, D, &c. it is known to what places of figures the area is exhibited accurate, and consequently this table does not want a table of corrections. As to the numeral coefficients they are far smaller, than the above table, and for that reason this is preferable, especially in a great number of ordinates.

Again, required the area of an hyperbola, whose ordinate is $\dfrac{1}{1+x}$.

For x write 0, $\tfrac{1}{4}$, $\tfrac{2}{4}$, $\tfrac{3}{4}$, $\tfrac{4}{4}$; and there will come out the five equidistant ordinates 1, $\tfrac{4}{5}$, $\tfrac{4}{6}$, $\tfrac{4}{7}$, $\tfrac{1}{2}$; therefore A will be $= \tfrac{4}{6}$, namely the middle ordinate; B $= \tfrac{4}{1050}$, namely the second difference of the three ordinates

nates $\frac{4}{7}$, $\frac{2}{7}$, $\frac{4}{7}$ in the middle; laftly, C$=\frac{1}{70}$, that is, equal to the laft difference of them all. And thefe values fubftituted in the expreffion for the five ordinates give $\frac{1}{7} + \frac{1}{111} + \frac{1}{960} = \frac{4567}{6760}$, which drawn into the bafe, or unity, and reduced into decimals, is .69317 for the area fought. And this area is at leaft exact in the fourth place of decimals, namely in which the firft figure of the laft term $\frac{1}{760}$ enters it.

But to find the differences more readily and more eafily, let a be the middle ordinate, b the fum of the two next to the middle one, and then c the fum of the two which follow, and fo on; then put

$$A = a,$$
$$B = b - 2A,$$
$$C = c - 2A - 4B,$$
$$D = d - 2A - 9B - 6C,$$
$$E = e - 2A - 16B - 20C - 8D,$$
$$F = f - 2A - 25B - 50C - 35D - 10E,$$
$$G = g - 2A - 36B - 105C - 112D - 54E - 12F,$$

And A, B, C, D, &c. will be the middle ordinate, the fecond difference of the three middle ones, the fourth of the five middle ones, and fo in the reft refpectively.

P R O P O S I T I O N XXXII.

LET a, b, c, d, e, f, &c. denote equidiftant terms continually tending to a ratio of equality, and the following equations will approximate to their relations.

$$2 \mid a - b = 0$$
$$3 \mid a - 2b + c = 0$$
$$4 \mid a - 3b + 3c - d = 0$$
$$5 \mid a - 4b + 6c - 4d + e = 0$$
$$6 \mid a - 5b + 10c - 10d + 5e - f = 0$$
$$7 \mid a - 6b + 15c - 20d + 15e - 6f + g = 0$$
$$8 \mid a - 7b + 21c - 35d + 35e - 21f + 7g - h = 0$$
$$9 \mid a - 8b + 28c - 56d + 70e - 56f + 28g - 8h + i = 0$$
$$10 \mid a - 9b + 36c - 84d + 126e - 126f + 84g - 36h + 9i - k = 0$$
&c.

This table is to be referved for ufe, to be confulted as often as is neceffary. And it appears that the numeral coefficients are the uncias of the powers of a binomial; and the demonftration is manifeft. For becaufe the terms are fuppofed continually to tend to the ratio of equality, their differences $a-b$, $b-c$, $c-d$, $d-e$, &c. will be fmall; then the differences

of

of the differences $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. will be less than the first differences; and the third $a-3b+3c-d$, $b-3c+3d-e$, &c. will be less than the second; and the fourth $a-4b+6c-4d+e$, &c. will be less than the third, and so *ad infinitum*. Therefore the first, second, third, &c. differences, put equal to nothing, as in the proposition, will continually approximate to the true relation of the terms. *Q. E. D.*

COROLLARY.

HENCE in a series of equidistant terms, if any term be wanting, it may be found by this Proposition. As if there be five terms a, b, c, d, e; their relation will be $a-4b+6c-4d+e=0$; and from this equation any one thereof will be given, as near as possible, from the rest being given.

And it is to be observed, that any term, *cæteris paribus*, is defined the more accurately, the nearer it stands to the middle of them all; and the errors from the true term are, as near as possible, as the numeral coefficients of the terms sought reciprocally. Therefore let the term sought stand either in the middle of them all, or as near to it as possible.

EXAMPLE.

REQUIRED the logarithm of the number 53 from the given logarithms of some foregoing numbers.

Put a for the logarithm sought; and it will be

$$l, 52=b=1.71600.33436$$
$$l, 51=c=1.70757.01761$$
$$l, 50=d=1.69897.00043$$
$$l, 49=e=1.69019.06800$$
$$l, 48=f=1.68124.12374$$
$$l, 47=g=1.67209.78579$$

The relation then between the seven terms a, b, c, d, e f, g, will be $a-6b+15c-20d+15e-6f+g=0$; therefore $a=6b-15c+20d-15e+6f-g$, where by substituting for b, c, d, e, f, g, their values, we shall have $1.72427.58726$ for a, or the logarithm of 53 sought, $.00000.00030$ being the error too much. But if there be given six logarithms, three of which stand before, and the other three after that which is sought; in that case, I say, the logarithm sought will be defined very accurately.

Therefore let A, B, C, D, &c. denote the sums of the given terms, which from each side are equally distant from that which is sought, and its values will be as in the following table:

A

$$
\begin{array}{c|c}
2 & \dfrac{A}{2} \\[1.5ex]
4 & \dfrac{4A-B}{6} \\[1.5ex]
6 & \dfrac{15A-6B+C}{20} \\[1.5ex]
8 & \dfrac{56A-28B+8C-D}{70} \\[1.5ex]
10 & \dfrac{210A-120B+45C-10D+E}{252} \\
& \&c.
\end{array}
$$

For example, let there be given the logarithms of the numbers 50, 51, 52, 54, 55, 56, required to find the logarithm of 53.　Put

$$A = l, \ 52 + l, \ 54 = 3.44839.71034$$
$$B = l, \ 51 + l, \ 55 = 3.44793.28656$$
$$C = l, \ 50 + l, \ 56 = 3.44715.80313$$

Then for A, B, and C, fubftitute thefe values in the expreffions for fix terms, and there will come out 1.72427.58695 for the logarithm of 53, the error being unity in the laft figure.　Hence in logarithmical, trigonometrical, aftronomical and other fuch like tables ; if there be wanting by chance any term, it may be inferted by this propofition ; or if you fufpect any term to be wrong, it may be corrected the fame way. For the expreffions here exhibited are general, as not at all depending on the nature of any particular table.

P R O P O S I T I O N XXXIII.

LET &c. ε, δ, γ, β, α, a, b, c, d, e, &c. denote the alternate terms in a feries running out both ways in infinitum, and if there be put $A = a + \alpha$, $B = b + \beta$, $C = c + \gamma$, $D = d + \delta$, $E = e + \varepsilon$, and fo on, the term ftanding in the middle between a and α will be equal to

$$
\begin{array}{l}
\tfrac{1}{2} \times A + \\[1ex]
\tfrac{1}{16} \times \overline{A - B} + \\[1ex]
\tfrac{1}{256} \times \overline{2A - 3B + C} + \\[1ex]
\tfrac{5}{2048} \times \overline{5A - 9B + C - D} + \\[1ex]
\tfrac{35}{65536} \times \overline{14A - 28B + 20C - 7D + E} + \\[1ex]
\qquad \&c.
\end{array}
$$

The numeral coefficients of the letters A, B, C, D, &c. are the differences of the uncias in different powers of a binomial ; and the coefficients

cients

cients which are drawn into the whole terms namely, $\frac{1}{2}$, $\frac{1}{16}$, $\frac{1}{256}$, $\frac{1}{2048}$, &c. are generated by the continual multiplication of the numbers $\frac{1}{4.1}$, $\frac{1}{4.4}$, $\frac{9}{4.6}$, $\frac{25}{4.8}$, &c. and thefe being known, the feries is produced at pleafure.

The feries alfo may be inveftigated by putting $z = 0$ in the fecond cafe of the twentieth Propofition, for then there will be had the ordinate or term which ftands in the middle of all.

But the terms of the feries may be collected into one fum, as you fee here,

$$2 \;\bigg|\; \frac{A}{2}$$

$$4 \;\bigg|\; \frac{9A - B}{16}$$

$$6 \;\bigg|\; \frac{150A - 25B + 3C}{256}$$

$$8 \;\bigg|\; \frac{1225A - 245B + 49C - 5D}{2048}$$

$$10 \;\bigg|\; \frac{39690A - 8820B + 2268C - 405D + 35E}{65536}$$

The firft expreffion is the firft term of the feries, the fecond is the fum of the firft and fecond, the third is the fum of the three firft terms, and fo on; wherefore from the alternate terms being given, the intermediate ones will be immediately given by this table, or by the feries. The firft expreffion $\frac{A}{2}$ fuffices when the fecond term of the feries is lefs than that which will enter the computation, and likewife in others; for the terms of the feries are the differences between the expreffions and the truth, as near as poffible; therefore we may always know what expreffion fuffices for our purpofe.

Thus, for example; fuppofe the logarithm of 53 be required from having thofe given of the numbers 46, 48, 50, 52, 54, 56, 58, 60; put

$$l,\ 52 + l,\ 54 = A = 3.44839.71035$$
$$l,\ 50 + l,\ 56 = B = 3.44715.40313$$
$$l,\ 48 + l,\ 58 = C = 3.44466.92309$$
$$l,\ 46 + l,\ 60 = D = 3.44093.90820$$

And thefe values being fubftituted in the feries, or in the expreffion, for eight terms, we fhall have 1.72427.58696 for the logarithm of 53; and in like manner we may find any other intermediate term. Therefore in the conftruction of tables it fuffices firft to feek fome terms at proper

distance

diſtance for the others may be inſerted by this method; for the terms firſt found are continually to be intercalated, until we come to the laſt which enters the table. And it is to be noted, that all the terms muſt be computed near the beginning of the table, by reaſon of their differences being great; then as the differences decreaſe we may ſkip the alternate terms, and afterwards, to every third, and to every ſeventh, when the differences are leſs. And this is the method that the great Sir *Iſaac Newton* hath laid down; but particular rules, deduced from the nature of the table to be conſtructed, are preferable, as theſe, for the moſt part, will perform the work with leſs labour.

F I N I S.

MISCELLANEA CURIOSA MATHEMATICA. Number VIII.

containing, A letter from Mr Bolton, A. M. shewing an easy construction and calculation of a problem for determining the places of a comet. A calculation of the solar eclipse, July 14, 1748, by Mr Machin's numbers in Brent's tables. By Mr Wm Brown. A calculation of that eclipse for London and Newcastle. By Mr R. Robinson. A table exhibiting the latitude and longitude where the central Annulus will be conspicuous. By Mr T. Cowper. Various solutions to the questions in No. VII. particularly a compleat solution of that important problem, Quest. 114, (founded on a scholium in Sir Isaac Newton's Principia) in its various hypotheses; with an investigation of the three following questions by Mr Emerson's method of Fluxions. By Mr Holliday. A collection of new questions to be answered in No. IX.

Number I. contains, among other useful matters, a discourse on the velocity and forces in bodies in motion; theorems for determining the sun's parallax; an universal spherico-catoptric theorem; a theorem on increments; of the maxima and minima in the celestial motions; to find the sun's horizontal parallax by the transit of Venus over the Sun's disk in 1761; a collection of curious questions in the present year's diaries, and the Gentleman's Magazine.

Number II. Solutions to all the questions in number I. an essay concerning the sums of the powers of an arithmetical progression; solutions to the questions in the Gentleman's Diary; and twenty new questions to be answered, &c.

Number III. contains solutions of questions in number I. and II. and in the Gentleman's Diary; also new questions proposed; the demonstration of four theorems in the Gentleman's Magazine, by Mr Facio; tables of chances for any number of points thrown on 1, 2, 3, &c. to 12 dice, inclusive, with the method of computation, &c. &c.

Number IV. An essay on the properties of triangles inscrib'd in, and circumscrib'd about two given circles. By Mr W. Chapple. A calculation at large of the lunar eclipse, that happen'd August 19, 1746, by Thomas Sparrow. A curious solution to question 131 in the Ladies Diary, by Mr Edw. Hauxley. Solutions to the questions in number III. with tables of sines and cosines, &c. also a solution to an exponential problem; by Mr Edw. Hauxley. A collection of new questions to be answered in number V. An easy, familiar and expeditious method for finding fluents, with examples to all Mr Emerson's forms, designed for beginners. Two new measures subservient to mechanical practice, by G? Polham.

Number V. A letter from E. Hauxley, concerning a curious, astronomical problem, for finding the place of a comet, &c. Observations at Cambridge on the late comet. By S. B. A letter from Philo-Whimsicolos, concerning an octave on the monochord, divided by geometrical proportionals. A table showing some properties of regular polygons. By Polygonus. The quadrature of the figures of tangents and secants, by the hyperbolic logarithms. Solutions to the questions in numb. V. A collection of new questions to be answered in number VI.

Number VI. An essay on fluxions, wherein the first principles are clearly demonstrated in a very easy and concise method, adapted to beginners. A general rule shewing how to solve all manner of simple fluxionary exponential problems; by E. Hauxley. Solutions to the questions in number V. A collection of new questions to be answered in No. VII. A very easy and familiar method of finding fluents, according to Mr Emerson's forms. Some observations on the late comet, by S. B. A table on the properties of regular solids. By Polyedra.

No. VII. An essay on the calendar, with regard to the correction of the Julian and Gregorian accounts in the celebration of Easter; By H. The theory of gunnery. By Mr Robert Robinson. A table of gunnery. Solutions of questions No. VI. A collection of new questions to be answered in No. VIII. The method for finding fluents, according to Mr Emerson's form, continued.